RESEARCH STATEMENT

BIANCA THOMPSON

1. Introduction

Cycle structures represent a steady state in a sea of chaos. My research focuses on iterating objects and asking about the cycle structures that form. Much of modern dynamics is based on foundational work by Julia and Fatou exploring discrete dynamical systems in [3, 4, 5, 12] . The field saw a resurgence of interest in the 1970s with the advent of computing technology. Since dynamical systems change over time with the next state depending on the current state, computing changed which examples are possible to generate that led us to new questions we can ask. The Mandelbrot set, iterations of the polynomial $z^2 + c$ starting at 0, is one such example that can be visualized and generated today. This set remains a source of research questions for those working in complex dynamics.

My field, arithmetic dynamics, is a relatively new and active field that also has emerged and grown because of computing power. Arithmetic dynamics is a subfield of number theory. In this field, known theorems in number theory are translated into an arithmetic dynamics setting. The advantage of this framing is that moving from elliptic curves, which have a lot of structure, to rational maps which have very little structure can give us a way to tackle a big topic in number theory; understanding the structure of the absolute Galois group $\mathrm{Gal}\,\overline{\mathbb{Q}}/\mathbb{Q}$. Rational maps have less structure than elliptic curves, but do have a lot of tools available that makes problems approachable in this setting. Table 1 illustrates how to reframe a number theory question as an arithmetic dynamic question by replacing a number theory object with an arithmetic dynamics object. In Table 1 we do not know what the equivalent object to CM elliptic curves should be. My research is exploring possible objects that may be equivalent.

My research has primarily focused on four areas in which I use fundamental tools in elementary number theory, arithmetic dynamics, and Galois theory to prove results and provide evidence for conjectures and problems in my field. I will describe some research areas in more detail below: the study of families of twists of dynamical systems and uniform bounds on the number of rational preperiodic points for these families (Section 3); the study of dynamics over finite fields and proportions of strictly periodic points (Section 4); and my current research in the study of arboreal Galois representations that arise when looking at the backwards orbit of a point over finite fields (Section 5). The techniques used in Section 5 will lead to future results in Section 4 which is what encouraged me to branch into arboreal Galois trees. These skills have helped me to design undergraduate research projects that are interesting and accessible as described in Sections 6.1 and 6.2.

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Table 1. This table illustrates how to reframe a number theory question as an arithmetic dynamic question by replacing a number theory object with an arithmetic dynamics object.

Number theory object	$\leftrightarrow$	Dynamical object
Elliptic curves	$\leftrightarrow$	Rational maps
Torsion points	$\leftrightarrow$	Periodic points
CM elliptic curves	$\leftrightarrow$	WE DON'T KNOW WHAT GOES HERE
$\mathbb{Z}$ or $\mathbb{Q}$ subgroups	$\leftrightarrow$	Forward or backwards orbits

2. Preliminaries

Arithmetic dynamics concerns itself with a set S and self mapping function $\phi : S \to S$. This allows for iteration

$$\phi^n = \underbrace{\phi \circ \phi \circ ... \circ \phi.}_{n \text{ terms}}$$

One goal of arithmetic dynamics is to classify a point α according to its behavior in the orbit defined to be $\{\phi^n(\alpha) : n \geq 0\}$. Often our questions stem from results in number theory restated in an arithmetic dynamics setting, such as problems from work by Serre in number theory on non CM elliptic curves in [24].

For the remainder, we take ϕ to be a morphism and $S = \mathbb{P}^N(K)$ where K is a field and $\mathbb{P}^N$ is projective N. In the case of $N = 1$, we may write $\phi = \frac{f}{g}$, a rational map where f and g have no common zeros. The degree of ϕ is $d = \max\{\deg(f), \deg(g)\} > 1$.

We define $\alpha \in \mathbb{P}^N(K)$ to be *periodic* if $\phi^n(\alpha) = \alpha$ for some $n \geq 1$, the smallest such n is called the *exact period of* α. The point α is *preperiodic* if some iterate $\phi^m(\alpha)$ is periodic. If α is not preperiodic, then we call α a *wandering point.*

Another important concept is *critical points* or *ramification points.* The point $\alpha \in \mathbb{P}^N(K)$ is a *critical point* for ϕ if the induced map on the tangent space T_α is 0. For $N = 1$, $\alpha < \infty$, and $\phi(\alpha) < \infty$, this is just the usual definition that $\phi'(\alpha) = 0$. In order to compute the derivative at the excluded points, we make a linear change of variables. We define the *critical orbit* of α to be $\{f^n(\alpha) : n \geq 1\}$. A guiding principle in complex dynamics is that the orbits of critical points are closely tied to the behavior of the dynamical system.

3. Twists

In this section K is a number field. We concern ourselves with morphisms

$$\phi : \mathbb{P}^N \longrightarrow \mathbb{P}^N$$
$$[x_0 : x_1 : ... : x_N] \mapsto [f_0(\mathbf{x}) : f_1(\mathbf{x}) : ... : f_N(\mathbf{x})]$$

where $\mathbf{x}$ is the $N + 1$-tuple $[x_0 : x_1 : ... : x_N]$.

Definition 3.1. We define

$$\mathrm{Hom}_d^N(K) = \{\phi : \mathbb{P}^N(K) \to \mathbb{P}^N(K) \colon \phi \text{ is a morphism of degree } d\}.$$

That is, ϕ is defined in each coordinate by homogeneous polynomials of degree d with coefficients in K. (We follow the convention that Hom_d^N refers to $\mathrm{Hom}_d^N(\overline{K})$, and similarly for PGL_{N+1}, where $\mathrm{PGL}_{N+1}(K)$ is the projective linear group over field $K = \mathrm{Aut}(\mathbb{P}^N)$.)

Definition 3.2. Let $\phi, \psi \in \mathrm{Hom}_d^N(K)$. We say the morphisms are *conjugate* if there is some $f \in \mathrm{PGL}_{N+1}\left(\overline{K}\right)$ such that $\phi^f = \psi$. They are *conjugate over K* if there is some $f \in \mathrm{PGL}_{N+1}(K)$ such that $\phi^f = \psi$.

Definition 3.3. For a map $\phi \in \mathrm{Hom}_d^N(K)$,

$$\mathrm{Twist}(\phi/K) = \left\{ \begin{array}{c} K\text{-equivalence classes of maps } \psi \in \mathrm{Hom}_d^N(K) \\ \text{such that } \psi \text{ is } \overline{K}\text{-equivalent to } \phi \end{array} \right\}.$$

An element $\psi \in \mathrm{Twist}(\phi/K)$ is called a twist of ϕ.

Example 3.4. *Let*

$$\phi(z) = z - \frac{2}{z} \text{ and } \psi(z) = z - \frac{1}{z}.$$

Also let $f(z) = z\sqrt{2}$. One may check that $\phi^f(z) = \psi(z)$. So ψ is a (quadratic) twist of ϕ. Solving $\phi^2(z) = z$ gives the $\mathbb{Q}$-rational two-cycle $\{\pm 1\}$. But ψ does not have rational points of exact period 2; solving $\psi^2(z) = z$ gives $\{\pm 1/\sqrt{2}\}$.

We denote $\mathrm{PrePer}\left(\phi, \mathbb{P}^N(K)\right) = \{P \in \mathbb{P}^N(K) \colon P \text{ is preperiodic under } \phi\}$. Morton and Silverman proposed the following conjecture in 1994:

Conjecture 1. *Let $K/\mathbb{Q}$ be a number field of degree D, and let $\phi : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree $d \geq 2$ defined over K. There is a constant $\kappa(D, N, d)$ such that*

$$\# \mathrm{PrePer}(\phi, \mathbb{P}^N(K)) \leq \kappa(D, N, d).$$

This implies uniform boundedness for torsion points on abelian varieties over number fields (see [2]). In the special case $N = 1$ and $d = 4$, the conjecture implies Merel's uniform boundedness of torsion points on elliptic curves [17]. Much work has been done on this problem, but only non-uniform bounds are known to date.

We can consider interesting families of dynamical systems, like $f_c = z^2 + c$, and see if Conjecture 1 will hold. In fact, Poonen conjectures in [23] that there is a precise bound over $\mathbb{Q}$ for this one parameter family:

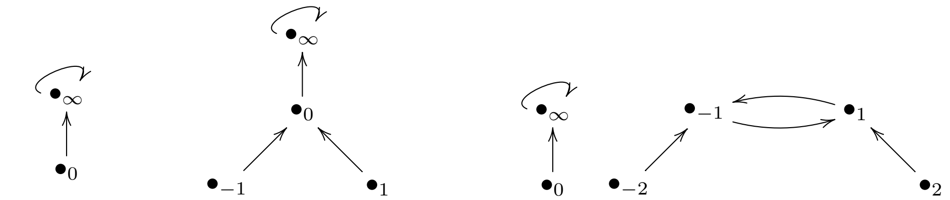
Conjecture 2. *If $z_0, c \in \mathbb{Q}$ such that z_0 has exact period n for $f_c(z) = z^2 + c$, then $n \leq 3$.*

Here the exact period n is bounded, whereas the Morton-Silverman conjecture bounds κ, the number of K-rational preperiodic points. Poonen shos that if the conjecture holds, then $\kappa = 9$ for quadratic polynomials. Even this refinement of the Morton-Silverman conjecture remains open. Morton [18] has shown that $n \neq 4$. Flynn, Poonen, Schaefer [6] showed that $n \neq 5$. And Stoll [27] proved that the Birch and Swinnerton-Dyer Conjecture implies $n \neq 6$. Given the difficulty of the question for quadratic polynomials, my Theorem 3.5 [14] regarding a different one-parameter family of quadratic functions appears surprisingly strong.

Theorem 3.5. *[Theorem 4.1 from [14]] The rational map $\phi_b(z) = \frac{z^2+b}{z}$ where $b \in \mathbb{Q}$ has no rational points with exact period $n \geq 5$.*

Using this theorem and prior results from Manes [15], I can describe all possible rational preperiodic structures for the family ϕ_b. See Figure 1.

It turns out that these two one-parameter families are different in a dynamically significant way. The quadratic functions ϕ_b considered above are all *quadratic twists* of the function $\phi(z) = \frac{z^2+1}{z}$. This is not the case for the quadratic polynomials f_c considered by Poonen and others.



(a) $b = 1$ (b) $b = -1$ (c) $b = -2$

FIGURE 1. All possible rational preperiodic graphs for $\phi_b(z) = \frac{z^2+b}{z}$.

Inspired by my proof of Theorem 3.5, Levy, Manes, and I sought a generalization to families of twists of dynamical systems in arbitrary degree and arbitrary dimension, more akin to the Morton-Silverman conjecture 1. We were able to prove the following:

Theorem 3.6. *[Theorem 2.9 from [14]] Let K be a number field and let $\phi \in \mathrm{Hom}_d^N(K)$. Then there is a uniform bound B_ϕ such that for all $\psi \in \mathrm{Twist}(\phi/K)$,*

$$\#\,\mathrm{PrePer}(\psi, \mathbb{P}^N(K)) \leq B_\phi.$$

We bounded the degree of the field extension needed to twist from ϕ to ψ. Then we were able to apply Northcott property to conclude the number of K-rational preperiodic points for twists of ϕ is bounded. The techniques used in the proof are similar to those used by Silverman in his result for abelian varieties in [25].

4. Finite Field Statistics

When iterating a polynomial function ϕ over a finite field, the orbit of any point $\alpha \in \mathbb{F}_{p^n}$ is a finite set. That is, all points are preperiodic, meaning the orbit eventually enters a cycle. Many natural questions about the structure of orbits over finite fields remain. Manes and I focus on the following question:

Question 1. Fix a polynomial: How does the proportion of periodic points in $\mathbb{F}_{p^n}$ vary as $n \to \infty$?

Manes and I answer the question in the special case that the polynomial map $\phi(z)$ can be viewed as an endomorphism of an underlying algebraic group. This restriction makes the structure of the periodic points particularly straight forward and is therefore a natural place to begin a more complete investigation of the question. We fix the notation $\mathrm{Per}\left(\phi, \mathbb{P}^N(K)\right) = \{P \in \mathbb{P}^N(K) \colon P \text{ is periodic under } \phi\}$.

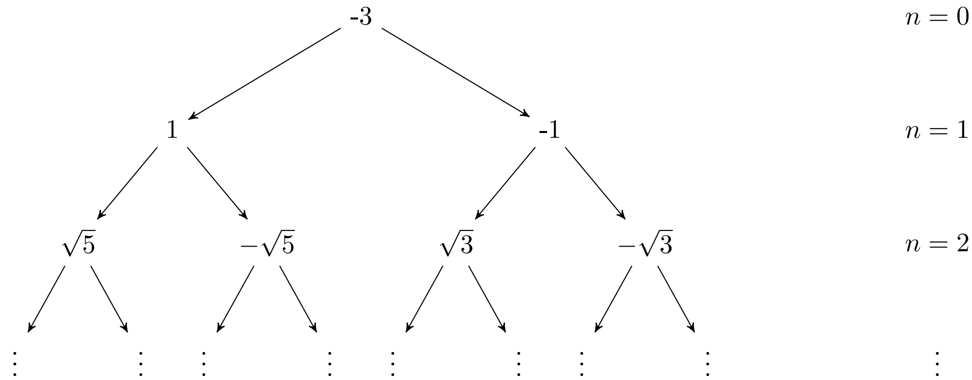
The naïve limit

$$\lim_{n\to\infty} \frac{\#\,\mathrm{Per}\left(\phi, \mathbb{F}_{p^n}\right)}{p^n}$$

does not exist in general because the map ϕ acts as a permutation polynomial whenever n is relatively prime to the multiplicative order of p modulo the degree of ϕ. However, we are able to find limiting proportions along towers of finite fields $\mathbb{F}_{p^n}$ with suitable divisibility conditions on n. We have the following two results for q an odd prime and similar results hold in the case $q = 2$ and for maps of composite degree:

Theorem (Theorems 4.5 and 5.7 in [16])**.** *Fix a prime p and let q be a different odd prime. Define δ to be the multiplicative order of p modulo q and $\mu = v_q(p^\delta - 1) \geq 1$.*

Figure 2. The infinitely rooted 2-ary tree representing the backwards orbit of $\alpha = -3$ for the function $\phi(z) = z^2 + 3$ over $\mathbb{F}_7$.



Let $f(z) = z^q$, and let $T_q(z)$ be the q^{th} Chebyshev polynomial. Then we have the following for any integer $\nu \geq 0$:

$$\lim_{\substack{n\to\infty \\ \delta | n \\ v_q(n)=\nu}} \frac{\#\operatorname{Per}(\phi, \mathbb{F}_{p^n})}{p^n} = \frac{1}{q^{\mu+\nu}}, \text{ and}$$

$$\lim_{\substack{n\to\infty \\ \delta | 2n \\ v_q(n)=\nu}} \frac{\#\operatorname{Per}(T_q, \mathbb{F}_{p^n})}{p^n} = \frac{q^{\mu+\nu}+1}{2q^{\mu+\nu}}.$$

where $v_q(n)$ is the q-adic valuation of n, that is $n = q^\nu d$, with $q \nmid d$.

The proofs take advantage of the fact that Chebyshev maps and power maps have a unique function in each degree, which we can write and iterate explicitly. Combining that explicit form with the defining equation for points in $\mathbb{F}_{p^n}$ leads to our result.

This paper is accessible to an undergraduate with a background in a topics in number theory course or abstract algebra and I am leading students on extending these results in Project 1 described in subsection 6.1.

5. Current research on finite fields and backwards orbits

Of more recent interest, like in [1, 8, 11, 13], is the backwards orbit of a point:

$$\{\phi^{-n}(\alpha) : n \geq 0\}.$$

We can visualize backward orbits of α under successive iterates of ϕ with an infinite rooted d-ary tree, T_∞. The set of vertices of this tree are

$$\bigsqcup_{n\geq 0} \phi^{-n}(\alpha) \subseteq \mathbb{P}^1(K^{\text{sep}})$$

whose edges are given by the action of ϕ (we take $\phi^0(\alpha) = \alpha$), see Figure 5. The absolute Galois group $\operatorname{Gal}(K^{\text{sep}}/K)$ acts on $T_\infty(\alpha)$, which we can think of as the *arboreal Galois representations* [1]. Their study dates back to work of R. W. K. Odoni in the 1980s [19, 20, 21].

A motivating problem of arboreal Galois representations of f is:

Problem 1. For any field K, for which rational functions $\phi \in K(x)$ do we have a finite index for $[\mathrm{Aut}(T_\infty) : G_\infty(\phi, \alpha)]$? Here $\mathrm{Aut}(T_\infty)$ is the collection of automorphisms on the infinite rooted d-ary tree T_∞ and $G_\infty(\phi, \alpha) = \varprojlim \mathrm{Gal}(K(f^{-n}(\alpha))/K)$.

Jones discusses in [10] the cases for which we can answer Problem 1. Such problems stem from work by Serre in number theory on non CM elliptic curves [24] as well as the goal of understanding the structure of the absolute Galois group $\mathrm{Gal}\,\overline{\mathbb{Q}}/\mathbb{Q}$.

I am focused on answering the following question:

Question 2. Fix a quadratic polynomial over $\mathbb{F}_p$: Consider the infinitely rooted binary tree T_∞ created when looking at the backwards orbit of α, where α is not periodic or in the critical orbit. Does there exist a sequence of nodes $t_0, t_1, \ldots$ in the tree such that $f(t_n) = f(t_{n-1})$ and $\mathbb{F}_p(t_n)/\mathbb{F}_p(t_{n-1})$ is nontrivial for all n?

Question 2 is motivated by Problem 1. Most results about arboreal Galois groups rely on finding primes that ramify and getting enough ramification to show that the Galois group of the tree has to grow [10]. Our idea is to show that the residue field has to grow at each level. If we could prove this for most primes then we could say something meaningful about $G_\infty(\phi)$ over a global field.

Jones and Boston discussed in [11] a condition we can check to determine if a map will have infinitely many Galois extensions in a row:

Theorem 5.1 (Proposition 2.3 [11])**.** *Let K be a finite field with characteristic not equal to 2 and $\phi \in K[x]$ with critical point γ. Then all iterates of ϕ are irreducible if and only if its adjusted critical orbit,*

$$\{-\phi(\gamma)\} \cup \{\phi^i(\gamma) : i = 2, 3, \ldots\},$$

contains no squares.

When restricted to $\mathbb{F}_p$ with $p > 2$ we can translate this theorem to mean the following.

Theorem 5.2. *Fix ϕ and suppose its critical orbit has length m. Then a function ϕ will have infinitely many nontrivial extensions in a row if it has m nontrivial extensions in a row.*

Benedetto, Juul, and I use Theorem 5.2 were able to answer part of Question 2 for some polynomials of the form $\phi(z) = z^2 + c$ over $\mathbb{F}_p$. We have shown

Theorem 5.3 (Benedetto, Juul, T.)**.** *For $\phi = z^2 + c$ with a critical orbit $\{0, c\}$, $\{0\}$, or $\{0, c, \phi(c)\}$ where $\phi(c)$ is fixed there exist a sequence of nodes $t_0, t_1, \ldots$ in this tree such that $f(t_n) = f(t_{n-1})$ and $\mathbb{F}_p(t_n)/\mathbb{F}_p(t_{n-1})$ is nontrivial for all n.*

6. Future Projects

I have several ideas for future work, including some collaborations underway. An undergraduate project about pursuing a natural extension of Manes and my results from Section 4 as well as going in a different direction with finite field statistics (Subsection 6.1 and Subsection 6.2). Through collaborations started at several workshops including ICERM and AIM, Benedetto, Juul, and I are working to extend our current results from Section 5 (Subsection 6.3)

6.1. **Project 1: Undergraduate Project on Finite Field Statistics.** Chebyshev polynomials arise by restricting the power map z^n to the quotient of $\mathbb{P}^1$ by the finite group of automorphisms $\{z, z^{-1}\}$. Similarly, quotients of elliptic curves lead to rational maps called *Lattès maps*. A natural next step in studying the proportions of periodic points in finite fields would be to consider the Lattès maps.

Definition 6.1. A rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d \geq 2$ is called a *Lattès map* if there is an elliptic curve E, a morphism $\psi : E \to E$, and a finite separable covering $\pi : E \to \mathbb{P}^1$ such that the following diagram is commutative:

$$\begin{array}{ccc} E & \xrightarrow{\psi} & E \\ \pi\downarrow & & \downarrow\pi \\ \mathbb{P}^1 & \xrightarrow{\phi} & \mathbb{P}^1 \end{array}$$

Example 6.2 (In [26])**.** *Let $E : y^2 = x^3 + ax + b$ be an elliptic curve. The the classical formula for $x(2P)$ and the isomorphism $x : E/\{\pm 1\} \to \mathbb{P}^1$ yield the Lattès map*

$$\phi(x) = x(2P) = \frac{x^4 - 2ax^2 - 8bx + a^2}{4x^3 + 4ax + 4b}.$$

Here ψ is the duplication map $\psi(P) = [2]P$, and the projection map π is given by $\pi(P) = \pi(x, y) = x$.

These maps have an underlying algebraic group structure, like the power maps and Chebyshev polynomials. In the case of Lattès maps, the group structure is that of elliptic curves.

Examples computed in SAGE suggest I can find the limit of strictly periodic points on Lattès maps that arise from supersingular elliptic curves. I would work with undergraduates to answer Question 1 for Lattés maps. They would need a background in algebra and elementary number theory and would use the programs I made in Sage as well as read [16] to solve the Lattès problem.

Power maps and Chebyshev polynomials have a function in each degree; Lattès maps do not share this property. This makes working with them more difficult. In order to study the behavior of their periodic points in a finite field we would not be able to use prior techniques from the power map and Chebyshev case. Since it has this added complexity, learning the proportion of periodic points would be a more interesting result. It may give insight to further cases that do not have an underlying group structure.

6.2. **Project 2: More Finite Field statistics.** Instead of asking Questions 1 we can ask:

Question 3. Fix a polynomial: How does the proportion of periodic points in $\mathbb{F}_p$ vary as $p \to \infty$?

Question 3 has been explored in [13] and [9]. Using techniques introduced by Odoni, they have shown you can use arboreal Galois trees to find proportions of periodic points. It is possible such techniques will extend to Question 1 for ϕ without an underlying group structure. I have been working with Juul and Benedetto to better understand the arboreal Galois techniques. We will then use these to answer Question 1 for more families of polynomials.Adding to these tools, Pink in [22] gave a description of the potential Galois group actions on T_∞ at each level of the tree. Using his descriptions we can calculate the proportion of periodic points of $z^2 - 1$ as $p \to \infty$, the last degree 2 case unanswered. I would gain insight on how to answer Question 3 for higher degree polynomials that were not covered by [9] and [13] as this proof would not rely on the ramification of primes as their results do.

6.3. **Project 3: Arboreal Galois Trees.** Expanding on work with Jones in [11], Boston, Goskel, and Xia found that when using a Markov chain process to build backwards orbit maps of different polynomials over finite fields that certain tree structures never appeared [7]. Benedetto, Juul, and my work suggests that those same tree structures that did not appear in Boston's work should not appear. We would formalize the reasoning using ideas developed at the AIM workshop Galois Theory of Orbits this past May. With this reasoning we would be able to expand Theorem 5.3 to more general critical orbits.

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