Lecture 10– Oct 2: Unions and intersections of sets, Cantor Set, Intro to Compact Sets.

Learning Goals

- Prove that the complement of an open set is closed.
- State properties of infinite unions and intersections of closed and open sets.
- Define the Cantor Set.
- Know the properties of the Cantor set.
- Define covers and subcovers and use these to define compact sets. Recognize the usefulness of compact sets.
- Be able to prove when a set is Not compact.
- Prove a property of compact sets.

Exam Wednesday.

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1 Loose ends from metric space section

Recall:

Theorem 1.1. *E is open if and only if E^c is closed.*
The complement of an open set is closed.

Recall that $E^c = \{p : p \in X \setminus E\}$, the complement of E in the universal set X.

Proof. We will write down equivalent statements to being open and arrive at the fact that E^c is closed.

E is open if and only if for all $x \in E$ x is an interior point of E. (This is the definition of open.)

□

Recall some set definitions:

1.
$$\bigcup_{i=1}^{n} A_i$$

2.
$$\bigcup_{i=1}^{\infty} A_i$$

3.
$$\bigcup_{\alpha\in\lambda} A_\alpha$$

Theorem 1.2. *Let $\{E_\alpha\}_{\alpha\in\lambda}$ be a (finite or infinite) collection of sets E_α. Then*

$$(\bigcup_{\alpha\in\lambda} E_\alpha)^c = \bigcap_{\alpha\in\lambda} (E_\alpha)^c.$$

*This is often referred to as **DeMorgan's Law.***

End of recall.

Theorem 1.3. *1. Arbitrary union of open sets is open.*

2. Arbitrary intersection of closed sets is closed.

3. Finite intersections of opens sets is open.

4. Finite union of closed sets is closed.

Question: Is an arbitrary intersection of open sets open?

Question: Is an arbitrary union of closed sets closed?

Fun (and useful) fact:

Theorem 1.4. *Say X is the universe. Then E is dense in X iff $\bar{E} = X$.*

2 New stuff: The Cantor Set

Here is an example of a set that we may find useful as a common counter example.

Example 2.1. The Cantor Set is defined to be the result of an infinite iterative process. Let $A_0 = [0, 1]$. We define $A_1 = [0, 1]$ with the middle third deleted. So $A_1 = [0, 1/3] \cup [2/3, 1]$.

$A_n = A_{n-1}$ with the middle thirds deleted from each line segment.

The **Cantor set** is defined to be

Properties of the Cantor Set (we'll define what some of these things are in the future).

- ______________________
- ______________________
- **Perfect**, ____________________________
- **Nowhere dense:** ____________________________
- ______________________
- ______________________

3 Compact sets

We've seen that finite sets are pretty nice to work with. Finite sets and finite operations don't break Calculus.

Finite sets contain their supremum. They are bounded. They are closed. Processes on finite sets have an end.

Unfortunately, most sets are not finite.

So we look at a set that's the next best thing: Compact Sets.

But first!

Definition 3.1. An **open cover** of E in X is a collection $\{G_\alpha\}_{\alpha \in \lambda}$ of open sets in X whose union contains E.

Example 3.2.

1. Let $E = [1/2, 1)$.

2. Let $E = (1, 2)$.

   Do we need all the G_n to cover E?

   Do we need infinitely many G_n to cover E?

   Is $\{G_n\}$ the only open cover of E?

3. See picture below:

Definition 3.3. A **subcover** of a cover $\{G_\alpha\}$ is a subcollection of the cover that still covers E. We may denote the subcover as $\{G_{\alpha_\gamma}\}$.

Question: Does every cover of E have a finite subcover?

Spaces that have finite subcovers are special. Now we can define our space that is almost as good as being finite.

Definition 3.4. A set K in X is **compact** in X if every open cover of K has a finite subcover.

Remark 3.5. Notice in order to be compact, EVERY open cover has a finite subcover. This is a for all statement.

If we wanted to prove a sett was NOT compact we would need to find one infinite cover that did not have a finite subcover. This is a there exists statement.

Example 3.6. Examples of spaces that are compact:

1. Finite sets are compact.

2. What else is compact?

Example 3.7. Examples of spaces that are not compact:

1. The subset $(1, 2) \subset \mathbb{R}$

2. $\mathbb{Z}$ is not compact. We only need to find one open cover with no finite subcover to prove it is NOT compact.

What else is compact? We need some theorems.

Theorem 3.8. *If K is compact then K is bounded.*

Recall E is **bounded** if there exists $M \in \mathbb{R}$ and a point $q \in X$ such that $d(p, q) < M$ for all $p \in E$. Alternatively, E is bounded if $E \subset B(x, r)$ where $x \in E$ and r is finite.

Proof. Suppose K is compact. Then every open cover of K has a finite subcover. Let's make a cover of K.

□