Lecture 11–Oct 9; Compact Set Properties

Learning Goals

- Be able to use the definition of compact to prove properties of compact spaces.
- Define and prove what it means for a subset to be open relative a bigger set.
- Recognize that if a set is compact in a larger set then it is still compact in a subset.
- Prove that closed sets of compact sets are compact.

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1 Properties

Recall

Definition 1.1. A set K in X is **compact** in X if every open cover of K has a finite subcover.

Theorem 1.2. *If K is compact then K is bounded.*

Recall E is **bounded** if there exists a neighborhood $B(x, r)$ for some x and r such that $E \subset B(x, r)$.

Proof. Suppose K is compact. Then every open cover of K has a finite subcover. Let's make a cover of K:

□

Theorem 1.3. *If K is compact in X then K is closed.*

Let's write down a few fact:

A set is closed if every limit point of K is a point of K. So if we have a point that is not in the set, we want to show it it not a limit point.

The definition of compact is that every open cover has a finite subcover.

Here are two ways we can try to prove it:

Proof. By contradiction. We want to consider a limit point p of K if $p \notin K$ we claim we can construct a cover with no finite subcover.

□

Proof. Let's instead do this by definition of what it means to be closed. So we want to show if $p \notin K$ then p is NOT a limit point of K.

Tricky bit: For all $q \in K$ Let $U_q = B(q, r/2)$ and $V_q = B(p, r/2)$ for $r = d(p, q)$.

□

Let's see how subsets change some of our definitions of open, closed, compact, and metric spaces.

- Recall: If $Y \subset X$ a metric space, then Y is also a metric space under the same metric. Why?

- For $Y \subset X$,
  Ball in X: $B(p, r) = \{x \in X : d(x, p) < r\}$.
  Ball in Y: $B(p, r) = \{x \in Y : d(x, p) < r\}$.

- A set U is open in Y means every point of U is an interior point of U in Y.
  An example:

Theorem 1.4. *Let $E \subset Y \subset X$. E is open in Y if and only if $E = Y \cap G$ for some G open in X.*

*We say E is **open relative to** Y.*

Proof. (sketch) Here is the idea of the proof. Assume E is open in Y.

If I assume $E = Y \cap G$ then restric neighborhoods $N(e)$ in X to a neighborhood in Y. □

- Compactness is an intrinsic property. So compactness should not change if I look at a subset of X.

  Theorem 1.5. *If $Y \subset X$ then K is compact in Y iff K is compact in X.*

  Proof. Suppose K is compact in X. Then given a cover of K in Y by $\{U_\alpha\}$

  If we assume first that K is compact in Y start with an open cover $\{G_\alpha\}$ in X. How would you argue that there is a finite subcover of $\{G_\alpha\}$? The argument is similar. Try as an exercise.

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Theorem 1.6. *A closed subset B of a compact set K is compact.*

Proof. To show B is compact take an open cover $\{U_\alpha\}$ of B.

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