Lecture 12– Oct 11; Compact sets Continued

Learning Goals

- Be able to prove that closed nested intervals have nonempty intersection.
- Prove that the closed interval in $\mathbb{R}$ is compact.
- State and prove Heine Borel.
- Give examples of why Heine Borel fails for non Euclidean metric spaces.

HW 5 due Friday at noon as well as any HW 3 rewrites.

---

1 More about compact sets

Recall

Theorem 1.1. *A closed subset B of a compact set K is compact.*

Corollary 1.2. *F is closed, K compact then $F \cap K$ compact.*

Proof. If K is compact then K is closed by our theorem from lecture 10. □

Theorem 1.3. *Nested closed intervals $[a_n, b_n]$ in $\mathbb{R}$ have non empty intersection.*
More generally see Theorem 2.36 in Rudin, Cantor's Intersection Theorem.

Here is an example of a nested closed interval with a point in its intersection:

Nested means for $m > n$ $a_n \le a_m \le b_m \le b_n$.
This theorem is Not true for nested open intervals.

Example 1.4. Let $A_n = (0, 1/n)$ for $n \ge 1$.

Proof. Let $x = \sup\{a_i : i \in \mathbb{N}\}$ this exists because the set is bounded by b_1 and $\mathbb{R}$ has the lub property.

□

Now we can show that closed intervals are compact!

Theorem 1.5. $[a, b]$ *is compact.*

More generally any k-cell in $\mathbb{R}^k$ is compact, where a k-cell is a higher dimensional closed rectangle.

Proof. Suppose not. Then there exists a cover $\{G_\alpha\}$ with no finite subcover.

Subdivide and repeat!

$\square$

Theorem 1.6 (Heine Borel Theorem)**.** *In R or $\mathbb{R}^n$ with the Euclidean metric, a set K is compact iff K is closed and bounded.*

Proof. The forward direction is true in any metric place. Please show this.

If K is closed and bounded, we need to use the fact that we are in $\mathbb{R}^k$ with the Euclidean metric to show that K is compact.

□

Example 1.7. Example of Heine Borel in use and examples of it failing for other metric spaces.

- The cantor set in $\mathbb{R}$ with the Euclidean metric.

- Consider the discrete metric on $\mathbb{R}$.

- Space of bounded continuous functions on $\mathbb{R}$ with the sup metric is not necessarily compact if it is closed and bounded.

  Recall $d(f, g) = \sup_{x \in K} |f(x) - g(x)|$.

  For example:

  Let $(C[0,1]; d)$ be the space of continuous functions on $[0,1]$ with metric $d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$. Then

  $$S = \{f : |f(x)| \leq 1\}$$

  is closed and bounded, but not compact. Why?