Lecture 13–Oct 18: Compact sets and Connected sets

Learning Goals

- State and prove equivalent statements to being compact.
- Be able to use sequences to show compactness.
- Be able to state the prove Bolzano-Weierstrass Theorem.
- State the definition of connected sets and separated sets.
- Be able to prove that a set is connected or not connected.
- Define the topologists comb.

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1 Last of Compact Sets

Theorem 1.1. *K is compact if and only if every infinite subset E of K has a limit point in K. ($E \subset K$)*

Proof. Forward direction. Assume K is compact. If there is no point of K that is a limit point of

E then ____.

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Proof continued:

Backward direction. This is true for any general metric space (Rudin Problem 2.26). We will show this is true for $\mathbb{R}$ with Euclidean metric.

Assume every infinite subset E of K has a limit point $p \in K$. Prove K is closed and bounded and hence compact by Heine Borel.

□

Theorem 1.2 (Bolzano Weierstrass Theorem)**.** *Every bounded infinite subset of* $\mathbb{R}^k$ *has a limit point in* $\mathbb{R}^k$.

Proof. If E is bounded

□

Remark 1.3. Fact: K is compact if and only if any collection of closed subsets K_α that has finite intersection property will have non empty intersection. The finite intersection property states that any finite sub collection has non empty intersection. This fact generalizes our nested intervals theorem. See Rudin 2.36.

2 Connectedness

Definition 2.1. If $A, B \subset X$, a metric space we call A and B **separated** if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty.

Example 2.2. Examples of separated and not separated sets.

1. Consider $A, B \subset \mathbb{R}^2$

2. Consider $(0, 1)$ and $(1, 2)$ in $\mathbb{R}$.

3. Separated sets are disjoint. But disjoint sets are not necessarily separated.

4. **The Topologists comb** T: Let $A = [0, 1] \times 0$ and $B = \frac{1}{n} \times [0, 1]$.

   Are A and B separated?

Definition 2.3. We say E is **connected** if E is not the union of two nonempty separated sets.

Example 2.4.

1. Consider Example 2.2 (1).

2. Consider Example 2.2 (2).

3. The Topologists comb T: To check that T is connected we need to show there is no way to write $T = A \cup B$ with A and B separated for any sets A and B.

4. The cantor set is not connected.

5. Douady and Hubbard proved that the Mandelbrot set is connected. Mandelbrot had conjectured that it wasn't because when he zoomed in to the image it looked like it didn't connect. (Proof is beyond the scope of this class).

6. Let $E = \{(x, y) : x, y \in \mathbb{Q}\} \subset \mathbb{R}^2$.

   (Check).

7. $[a, b]$ is connected.

Theorem 2.5. *$[a, b]$ is connected. ($\mathbb{R}$ euclidean).*

Proof. If not then there exists a separation of $[a, b] = A \cup B$. WLOG say $b \in B$.

$\square$

Rudin shows that $E \subset \mathbb{R}$ is connected if and only if E has an 'interval like' property.

Theorem 2.6. *$E \subset \mathbb{R}$ is connected iff if $x, y \in E$ and $x < z < y$ then $z \in E$.*

Another characterization of connected: A set E is connected if and only if E is not the disjoint union of A and B where A and B are open relative to E. (and therefore closed in E).