Lecture 14–Oct 23; Sequences

Learning Goals

- Be able to prove that $[a, b] \subset \mathbb{R}$ is connected.
- Define sequences and what it means for a sequence to converge.
- Wrestle with the definition to understand the importance of each quantifier in the definition.
- State and prove some properties of convergent sequences.

Recall:

We say E is **connected** if E is not the union of two nonempty separated sets. we call A and B **separated** if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty.

Theorem 0.1. *$[a, b]$ is connected. ($\mathbb{R}$ euclidean).*

Proof. If not then there exists a separation of $[a, b] = A \cup B$. WLOG say $b \in B$.

Let $s = \sup A$ (which exists since we are in $\mathbb{R}$) then either $s \in A$ or s is a limit point of A because we know $s - \epsilon$ is not an upper bound so there exists $a \in A$ such that $s - \epsilon < a \leq s$. So $s \in \overline{A}$.

Since A and B are separated sets

□

1 Sequences in Metric spaces

We are now in Rudin Chapter 3.

Example 1.1.

- Let $X = C(\mathbb{R}) :=$ continuous bounded functions from $\mathbb{R}$ to $\mathbb{R}$. Let the metric be $d(f, g) = \sup_{x \in \mathbb{R}} |f(x) - g(x)|$.

  Consider

  Do these converge to some f?

- Convergent sequences examples

- Let $\mathbb{Q}$ have the p-adic metric. $d(x, y) = |x - y|_p = p^{-\alpha}$ where $x - y = p^\alpha b$ with $p \nmid b$.

  Consider $\mathbb{Q}$ with 5-adic metric. Look at the sequence $\{\frac{1}{5^n}\}$.

Recall

Definition 1.2. A **sequence** $\{p_n\}$ in metric space X is a function $f : \mathbb{N} \to X$ mapping $n \mapsto p_n$, a point in X.

Definition 1.3. The **range** of $\{p_n\}$ is the set $\{x : x = p_n \text{ for some } n\}$.

A sequence is **bounded** if its range is bounded.

Example 1.4. The sequence $p,\ p,\ p...$ has range

Definition 1.5. A sequence $\{p_n\}_{n\in\mathbb{N}}$ converges if there exists a $p \in X$ such that for all $\epsilon > 0$ there exists $N \in \mathbb{Z}^+$ such that $n \geq N$ implied $d(p, p_n) < \epsilon$.

We write $p_n \mapsto p$ or $\lim_{n\mapsto\infty} p_n = p$.

We say 'p_n converges to p' or 'p is the limit of $\{p_n\}$.'

To show p_n converges to p we must for each $\epsilon > 0$ find an N that makes p_n and p close together.

Activity 1 today: Below you will find several statements involving a sequence $\{a_n\}$ of real numbers and a real number $L>$ In each case, consider the statement as an "alternative" to the definition $\{p_n\} \mapsto L$. Provide an example of a sequence of real numbers and a number L that satisfies the "definition" and yet does no converge to L. Accompany your example with a verbal explanation of the inadequacies of the definition. Be prepared to talk about your example with the class.

1. The sequence $\{a_n\}$ converges to L if for all $\epsilon > 0$ there exists an $n \in \mathbb{N}$ such that $d(a_n, L) < \epsilon$.

2. The sequence $\{a_n\}$ converges to L if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for some $n > N$, $d(a_n, L) < \epsilon$.

3. The sequence $\{a_n\}$ converges to L if for all $N \in \mathbb{N}$ there exists $\epsilon > 0$ so that for all $n > N$, $d(a_n, L) < \epsilon$.

4. The sequence $\{a_n\}$ converges to L if for all $N \in \mathbb{N}$ and for all $\epsilon > 0$ there exists an $n > N$ such that $d(a_n, L) < \epsilon$.[1]

[1]This exercise is taken from *Closer and Closer: Introducing Real Analysis* by Carol S. Shumacher, Copyright Jones and Bartlett Publisher, 2008.

Activity 2: Determine if the following statements are true or false.

1. $p_n \mapsto p$ and $p_n \mapsto p'$ then $p = p'$.

2. $\{p_n\}$ bounded then p_n converges.

3. p_n converges then p_n is bounded.

4. $\lim_{n \mapsto \infty} p_n = p$ then p is a limit point of range of $\{p_n\}$

5. p is a limit point of $E \subset X$ then there exists a sequence $\{p_n\}$ in E such that $p_n \mapsto p$.

6. $p_n \mapsto p$ if and only if every neighborhood of p contains all but finitely many p_n.

Extra scratch page;

My justifications for 1-6 in Activity 2.

1. Suppose p_n converges to p and p_n converges to p'. Then because p_n converges to p for all $\epsilon > 0$ there exists and N such that for all $n \geq N$ $d(p_n, p) < \epsilon/2$. Similarly so there exists an N' such that for all $n \geq N'$ $d(p_n, p') < \epsilon/2$. Choose $\epsilon = d(p, p')$ and let $n \geq \max\{N, N'\}$ then

$$\epsilon = d(p, p') \leq d(p', p_n) + d(p, p_n) < \epsilon.$$

   This implies that $d(p, p') = 0$, done.

2. Sequence of $0, 1$ is bounded but does not converge.

3. Let $\epsilon = 1$. Since p_n converges to p there exists an N such that $n \geq N$ implies $d(p_n, p) < 1$. Let $R = \max\{1, d(p_1, p), ..., d(p_{N-1}, p)\}$.

   So all points are in $B(p, R+1)$. Hence they are bounded.

4. Let $p_n = p$.

5. Idea; For all $n \in \mathbb{N}$ choose $p_n \in B(p, 1/n)$. This point exists by definition of a limit point. We want to show p_n converges to p. Given $\epsilon > 0$ let $N = \lceil \frac{1}{\epsilon} \rceil$.

   Check that $n \geq \lceil \frac{1}{\epsilon} \rceil$ implies $\frac{1}{n} \leq \epsilon$. So $d(p_n, p) < \frac{1}{n} \leq \epsilon$, as desired.

6. backwards direction; for all $\epsilon > 0$ the neighborhood $B(p, \epsilon)$ contains all but $p_{i_1}, p_{i_2}, ...p_{i_r}$ for $i_1 < ... < i_r$.

   Then $n \geq i_{N+1}$ implies $d(p_n, p) < \epsilon$.

   Forward direction. Proof by picture. Look at a sequence of p_n converging to p. We can use an argument similar to 3. to show every neighborhood contains all but finitely many p_n.