Lecture 15– Oct 25; When Sequences converge

Learning Goals

- State and prove limit properties of convergent sequences.
- Be able to define a subsequence and what it means to be sequentially compact.
- Prove statements about convergent subsequences.
- Define Cauchy sequences and complete sets.
- Prove some properties of Cauchy sequences.

FRIDAY October 27: HW 7 DUE at **12pm**

Recall; A sequence $\{p_n\}_{n\in\mathbb{N}}$ converges if there exists a $p \in X$ such that for all $\epsilon > 0$ there exists $N \in \mathbb{Z}^+$ such that $n \geq N$ implied $d(p, p_n) < \epsilon$. The index N may depend on ϵ.

1 limit properties

Suppose we have sequences in $\mathbb{R}$ or $\mathbb{C}$ $\{s_n\}$ and $\{t_n\}$ where $s_n \mapsto s$ and $t_n \mapsto t$.

Theorem 1.1.

$$\lim_{n\mapsto\infty} s_n + t_n = s + t.$$

The limit of the sums is the sum of the limits.

Proof. IDEA Bound

$$|s_n + t_n - (s + t)| = |s_n - s + t_n - t| \leq |s_n - s| + |t_n - t|.$$

Now given $\epsilon > 0$

□

Theorem 1.2.

$$\lim_{n \mapsto \infty} cs_n = cs.$$

and

$$\lim_{n \mapsto \infty} s_n + c = s + c.$$

Proof. Bound $|cs_n - cs| = |c||s_n - s|$.

Bound $|s_n + c|$.

□

Theorem 1.3.

$$\lim_{n \mapsto \infty} s_n t_n = st.$$

Proof. Tricky!

$$|s_n t_n - st| = |(s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)|$$

Now finish the proof.

□

Rudin has more limit theorems. See Rudin.

2 Subsequences

Definition 2.1. Let $\{p_n\}$ be a sequence. Let $n_1 < n_2 < n_3...$ be increasing indices. Then $p_{n_1},\ p_{n_2}, ...$ is a **subsequence** of $\{p_n\}$.

Example 2.2. Let $\{p_n\} = \{1/2, 2/3, 3/4, 4/5, 5/6, ...\}$

Question: If a sequence converges must every subsequence converge? Guesses:

Answer:

Question; If a sequence does not converge does there exist a convergent subsequence?

Example 2.3. $1, \pi, 1/2, \pi, 1/3, \pi, ...$ We see this sequence does not converge.

Theorem 2.4. *In a compact metric space X every sequence has a convergent subsequence converging to a point of X. (When seeing compact think 'small').*

We say X is ***sequentially compact.***

Remark 2.5. X is compact iff X is sequentially compact.

Proof. Let R =range of $\{p_n\}$. If R is finite,

If R is infinite, then since X is compact R has a limit point p.

□

Corollary 2.6 (Bolzano-Weierstrass)**.** *Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.*

Proof. View a sequence as a subset of a large, closed disk (since it's bounded).

□

Lemma 2.7. *In $\mathbb{R}$ if a sequence is* ***monotonic*** *(always increasing or always decreasing) and bounded, then the sequence converges to its sup or inf.*

Proof. idea $s - \epsilon$ is not an upper bound so $s - \epsilon < p_n$.

□

3 Cauchy sequences

Definition 3.1. A sequence $\{p_n\}$ is Cauchy if for all $\epsilon > 0$ there exists an N such that for all $m, n \geq N$ then $d(p_n, p_m) < \epsilon$.

"Past index N points get close to each other."

Theorem 3.2. *If $\{p_n\}$ converges then $\{p_n\}$ is Cauchy.*

Proof. Idea Consider $d(p_n, p_m)$

□

NOTE; Not every Cauchy sequence converges.

Example 3.3. Consider the sequence in the rationals $\mathbb{Q} : \{3, 3.1, 3.14, 3, 141, ...\}$

Definition 3.4. A metric space X is **complete** if every Cauchy sequence in X converges to a point of X.

Fact: Compact metric spaces are complete. Also $\mathbb{R}^k$ is complete.