Lecture 16–Oct 30; Cauchy sequences and Completion

Learning Goals

- State and use equivalent definitions to being Cauchy.
- Know definition of diameter and be able to use it in a proof.
- Be able to define completion and know how to complete a metric space.
- Be able to use the definition of Cauchy to prove compact metric spaces are complete.

HW 8 due Friday, note problem 7 needs material from Wednesday to do. Exam next week; it will cover HW 1-8, with an emphasis on the new material.

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1 Cauchy Sequence continued

Recall:

Definition 1.1. A sequence $\{p_n\}$ is Cauchy if for all $\epsilon > 0$ there exists an N such that for all $m, n \geq N$ then $d(p_n, p_m) < \epsilon$.

"Past index N points get close to each other."

Definition 1.2. Let E be a nonempty subset of a metric space X and let S be the set of all real numbers of the form $d(p, q)$ with $p \in E$ and $q \in E$. The sup of S is the **diameter** of E. We denote this as diam E.

Remark 1.3. If $\{p_n\}$ is a sequence in X and E_N consists of the points $p_N, p_{N+1}, ...$ then $\{p_n\}$ is Cauchy iff

$$\lim_{N \mapsto \infty} diam\ E_N = 0.$$

Theorem 1.4. *1. If $\bar{E}$ is the closure of $E \subset X$, where X is a metric space, then diam $\bar{E}$ =diam E.*

2. If K_n is a sequence of compact sets in X such that $K_n \supset K_{n+1} \supset ...$ and if $\lim_{n \mapsto \infty} diam\ K_n = 0$ then

$$K := \bigcap_{n=1}^{\infty} K_n$$

consists of exactly one point.

Proof. IDEA

Of a:

Of b: We know that the intersection is non empty by generalizing our nested intervals theorem.

□

2 Complete Metric spaces

Recall:

Not every Cauchy sequence converges. But it is true for complete metric spaces.

Definition 2.1. A metric space X is complete if every Cauchy sequence in X converges to a point of X.

Question: What spaces are complete? Guesses?

Theorem 2.2. *Compact metric spaces are complete.*

Proof. Let $\{x_i\}$ be Cauchy in X.

□

Theorem 2.3. *Closed subset E of complete metric space is complete.*

Proof. IDEA Satrt with a Cauchy sequence $\{x_i\} \in E$.

□

Theorem 2.4. $\mathbb{R}^k$ *is complete.*

Proof. IDEA

□

Example 2.5. in $\mathbb{R}$. Does $x_n = 1 + \frac{1}{2} + ... + \frac{1}{n}$ converge?

Example 2.6. $\mathbb{Q}$ is not Cauchy (we showed this last time.) But, $\mathbb{Q}$ can be extended to $\mathbb{R}$.

Theorem 2.7. *Every metric space* (X, d) *has a completion* (X^*, Δ) *where* x^* *extends* X *and* Δ *extends* d.

How does it do this, you ask. Given X, let

$X^* = \{$____$\}$

Continued on the next page.

For P and $Q \in X^*$ let

$$\Delta(P, Q) = \rule{8cm}{0.4pt}$$

Then X is 'isometrically' (there exists a bijection with a subset of X^* that preserves distances) embedded into X^* via

and X is complete.

Example 2.8.

1. $\mathbb{Q}$ may be completed with respect to the Euclidean metric, and when we do this we get $\mathbb{R}$.
2. You can also complete $\mathbb{Q}$ with respect to the p-adic metric.

   This creates $\mathbb{Q}_p$, the p**-adic rationals**.