Lecture 17–Nov 1; lim sup, series, and Cauchy Criterion

Learning Goals

- Be able to define and find lim inf and lim sup of a sequence.
- Be able to use and prove properties of lim sup and lim inf .
- Know the values of special limits and be able to prove that is the correct limit.
- Formalize the definition of an infinite series, recognizing that past mathematicians had other iterpretations.
- State the Cauchy Criterion and be able to use it to prove a series is convergent or divergent.

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1 lim sup and lim inf

Definition 1.1. For a sequence $\{s_n\} \in \mathbb{R}$ let $E = \{x : s_{n_k} \mapsto x$ for some subsequence $s_{n_k}\}$. This set E contains all subsequential limits and may include $\pm\infty$.

Note

Definition 1.2. Let

$$s^* = \sup E := \limsup_{n \mapsto \infty} s_n.$$

be the upper limit of s_n
and

$$s_* = \inf E := \liminf_{n \mapsto \infty} s_n$$

be the lower limit of s_n.

Note,

Definition 1.3. Alternatively we may define s^* as follows:

$$s^* = \lim_{n \mapsto \infty} \sup_{k \geq n} s_k.$$

We can think of this definition as the limit of the supremum of the tails.

Example 1.4. Examples of liminf and limsup.

1. $s_n = \{.1, 3/2, .11, 4/3, .111, 5/4, ...\}$

2. Let $\{s_n\}$ be a sequence containing all rational numbers.

3. Suppose $s_n \mapsto s$.

Theorem 1.5. *Subsequential limits of $\{s_n\}$ in a metric space X form a closed subset of X.*

Proof. Direct proof. Use the definition of closed to show this. □

Theorem 1.6. *Let $\{s_n\}$ be a sequence in $\mathbb{R}$. Then s^* has the following properties.*

1. $s^* \in E$.

2. If $x > s^*$ then there exists N such that for all $n \geq N$ implies $s_n < x$.

More over s^* is the only number with these properties. An analogous statement holds for s_*.

Proof. IDEA To prove a: Treat $s^* = \pm\infty$ as separate case. For s^* finite, appeal to previous theorem as well as Rudin Theorem 2.28.

If $s^* = -\infty$ then E contains only one element $-\infty$ and no subsequential limit.

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Prove b using contradiction.

To prove uniqueness, assume not unique and arrive at a contradiction.

□

Theorem 1.7. *If* $s_n \leq t_n$ *for all* $n \geq N$ *then* $\limsup s_n \leq \limsup t_n$*. Equivalent statement holds for* $\liminf$*. If the sequence converges then* $\lim s_n \leq \lim t_n$*.*

Fact: If $0 \leq x_n \leq s_n$ for $n \geq N$ (where N is fixed) and if $s_n \mapsto 0$ then $x_n \mapsto 0$.

Special limits.

1. Let $p > 0$:

$$\lim_{n \mapsto \infty} \frac{1}{n^p} = 0.$$

2. Let $p > 0$:

$$\lim_{n \mapsto \infty} p^{1/n} = 1;$$

3.

$$\lim_{n \mapsto \infty} n^{1/n} = 1.$$

4.

$$\lim_{n \mapsto \infty} \frac{n^\alpha}{(1+p)^n} = 0.$$

5. For $|x| < 1$

$$\lim_{n \mapsto \infty} x^n = 0.$$

2 Series

Question: What does it mean that an infinite sum equals a finite number?

Consider:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ... = \frac{\pi^2}{16}.$$

Consider:

$$1 - 1 + 1 - 1 + 1 - 1 + ... =?$$

Past mathematicians knew that

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + ... = \frac{1}{1 - 1/3} = 3/2.$$

which is a special case of

$$1 + x + x^2 + x^3 + ... = \frac{1}{1 - x}.$$

So

Euler accepted that

We need to have one interpretation so we have consistency.

Definition 2.1. We define a finite series to be: Given by $\{a_n\}$ let $s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + ... + a_n$. We call this the n**th partial sum.**

Now $\{s_n\}$ is a sequence.

Definition 2.2. We may write the sequence of partial sums as $\sum_{k=1}^{\infty} a_k$, and we call it an **infinite series.**

Remark 2.3. Note that s is the limit of the sequence of partial sums and is not obtained simply by addition.

Question: When does a series converge?
Guesses:

Example 2.4. Let $a_n = 1/n$. The **harmonic series** $\sum_{n=1}^{\infty} 1/n$

In general for a series in $\mathbb{R}$ we may check if it converges using the Cauchy Criterion.

Theorem 2.5. *Cauchy Criterion*
$\sum_{k=1}^{\infty} a_k$ *converges iff for all* $\epsilon > 0$ *there exists an* N *such that for* $m, n \geq N$ *implies* $|\sum_{k=n}^{m} a_k| < \epsilon$.