Lecture 18–Nov 6; Series test!

Learning Goals

- Be able to state prove tests such as Cauchy criterion, Comparison test, geometic series, and the term test.
- Recognize when a series test is inappicable.
- Use some series tests to prove convergence or divergence.
- Prove e is irrational.

There is more on series than we will cover. Please see Rudin pg 72-75 for more about operations with series.

Exam is available on Sakai. Due Friday at 5pm outside my door.

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1 Convergence Tests

Recall

Definition 1.1. We may write the sequence of partial sums as $\sum_{k=1}^{\infty} a_k$, and we call it an **infinite series.**

We saw that an infinite series will converge if it fulfills the Cauchy criterion:

Theorem 1.2. *Cauchy Criterion* $\sum_{k=1}^{\infty} a_k$ *converges iff for all* $\epsilon > 0$ *there exists an* N *such that for* $m, n \geq N$ *implies* $|\sum_{k=n}^{m} a_k| < \epsilon$.

Corollary 1.3. *Term test* $\sum a_n$ *converges then* $\lim_{n \mapsto \infty} a_n = 0$.

Proof. Let $m = n$ in the Cauchy criterion.

□

Remark 1.4. If the terms in the series do not converge to 0 then the series diverges. The converse is **false**. If the terms converge to 0 the series may diverge.

Example 1.5. Example of term test not showing convergence:

Theorem 1.6. *Non-negative series. If $a_n \geq 0$ then $\sum a_n$ converges if and only if partial sums are bounded.*

Proof. Idea This follows from the fact that bounded monotonic sequences converge. □

Theorem 1.7. *Comparison Test.*

1. *If $|a_n| \leq c_n$ (eventually) for n large enough and $\sum c_n$ converges then $\sum a_n$ converges.*
2. *If $a_n \geq d_n \geq 0$ (eventually) for n large enough and $\sum d_n$ diverges then $\sum a_n$ diverges.*

Proof. Since the $\sum c_n$ converges, by Cauchy criterion we have the following. □

2 Geometric series

Theorem 2.1. *If $|x| < 1$ then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. Else the series diverges. We call this series the **geometric series.***

Proof. Note the partial sum is

$$s_n = 1 + x + x^2 + ... + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

□

Example 2.2.

$$\sum_{n=0}^{\infty} \frac{1}{n!}$$

Aside: In fact $\sum \frac{1}{n!}$ converges very quickly to e.

Theorem 2.3. *e is irrational.*

Proof. By contradiction. Suppose $e = \frac{m}{n}$.

□

Here's a new series;

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

Using our current tests, can we determine if this converges or diverges? Try:

Theorem 2.4. *If* $a_1 \geq a_2 \geq ... \geq 0$ *(monotonically decreasing, non negative) then* $\sum_{n=1}^{\infty} a_n$ *converges iff* $\sum_{k=0}^{\infty} 2^k a_{2^k}$ *converges.*

Proof. If we can show the sequence of partial sums forms a bounded sequence we could use theorem 1.6 from above to finish the proof.

Consider

$$s_n = a_1 + a_2 + a_3 + ...a_n$$
$$t_k = a_1 + 2a_2 + 2^2 a_4 + ... + 2^k a_{2^k}.$$

□

Theorem 2.5. $\sum \frac{1}{n^p}$ *converges if* $p > 1$ *and diverges if* $p \leq 1$.

Proof. Do this by cases:
If $p \leq 0$ then

If $p > 0$ consider $\sum_k 2^k \frac{1}{2^{kp}}$

□

Example 2.6.

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Theorem 2.7. *Root Test. Given* $\sum a_n$ *let* $\alpha = \limsup \sqrt[n]{|a_n|}$ *then*

1. $\alpha < 1$ *then the series converges.*
2. $\alpha > 1$ *then the series diverges.*
3. $\alpha = 1$ *then the test is inconclusive.*

Proof. Use comparison test with the geometric series.
If $\alpha < 1$,

If $\alpha > 1$

If $\alpha = 1$ notice $\sum 1$ diverges but $\sum \frac{1}{n^2}$ converges. So it is inconclusive. □

Theorem 2.8. *Ratio Test*

1. $\sum a_n$ *converges if* $\limsup \frac{a_{n+1}}{a_n} < 1$.
2. $\sum a_n$ *diverges if* $\limsup \frac{a_{n+1}}{a_n} > 1$ *for some* $n \geq N_0$. *(eventually bigger).*

Proof. Again use the comparison test with the geometric series.

□

Note the ratio test is easier, but the root test is more powerful. See Rudin 3.36 and 3.37.