Lecture 19–Nov 8; Absolute and Conditional convergence

Learning Goals

- Define power series and radius of convergence.
- Define e and use the definition to show equivalence to the limit definition of e.
- Define absolute convergence and conditional convergence and demonstrate the difference between each series.
- State and prove Riemann rearrangment theorem.
- Given two series, know how to do summation by parts.

There is more to series than we can touch upon in the class, see Rudin chapter 3 for more interesting facts and theorems!

Exam due Friday at 5pm. Hw 9 due Nov 17th at noon. Hw 10 due Nov 22nd in class. No office hours Thursday.

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1 Power Series

Definition 1.1. Let $c_n \in \mathbb{C}$. Then $\sum_{n=0}^{\infty} c_n z^n = c_0 + c_1 z + c_2 z^2 + ...$ is a **power series** in z a complex variable.

Example 1.2. $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$

This leads to the question: When does a power series converge?

Theorem 1.3. *Let $\alpha = \limsup \sqrt[n]{|c_n|}$. Let $R = \frac{1}{\alpha}$. Then if $|z| < R$ the series converges. If $|z| > R$ then the series diverges. This creates a disk which we call the* ***radius of convergence.*** *If $|z| = R$ then the test is inconclusive.*

Note

Proof Idea. Use the root test and look at $\limsup \sqrt[n]{|c_n|}$.

□

Example 1.4. $\sin x$ power series.

Now define

Definition 1.5. $e = \sum \frac{1}{n!}$

Theorem 1.6. $\lim_{n \mapsto \infty}(1 + \frac{1}{n})^n = e$.

Proof. Let

$$s_n = \sum_{k=0}^{n} \frac{1}{k!},$$

$$t_n = (1 + \frac{1}{n})^n.$$

□

Observations/Recall from calculus

- Power series behave well. Power series are 'continuous'.
- The sum of the derivative is the derivative of the sum.
- The integral of the sum is the sum of the integral.

Famous/cool series

- $f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \cos(12^n \pi x)$

- Taylor series, a type of power series. We'll see more of this later.
- Fourier series.

2 Absolute Convergence

Definition 2.1. $\sum a_n$ **converges absolutely** means $\sum |a_n|$ converges.

Example 2.2. $\sum \frac{(-1)^k}{k}$

Note!

Theorem 2.3. *If $\sum a_n$ converges absolutely then it converges.*

Proof Idea. Use the Cauchy criterion and look at 'tails' of series and bound them;

$$\sum_{k=n}^{m} a_k.$$

□

Example 2.4. Does $\sum |\frac{\cos n}{n^2}|$ converge? Does it converge absolutely?

3 Rearrangements

Question: If we rearrange terms of $\sum a_n = A$, must it still converge to A?
Guesses?

Example 3.1. Using the alternating harmonic series find a rearrangement of the series that gives

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Theorem 3.2. *Riemann's Rearrangement Theorem.*

1. *In a conditionally convergent series the sum of the positive terms is a divergent series (as is the sum of the negative terms.)*
2. *Let $\sum a_n$ converge conditionally, and let S be a given real number. Then there exists a rearrangement of $\sum a_n$ such that the series goes to S.*

Proof Idea of 2. Part 1 says that all the positive and negative terms diverge. So we can begin by adding positive terms until we a get a number larger than S.

□

Remark 3.3. Arrangements are not unique. You can construct many arrangements to get S. CHALLENGE! Write a program that can find a rearrangement of a series to get S! White might the algorithm look like?

Theorem 3.4. *If a series converges absolutely then every rearrangements has the same sum.*

4 Multiplying series

Question; What is $\sum a_n b_n$ if we know $\sum a_n$ and $\sum b_n$?

Let $A_n = \sum_{k=0}^{n} a_k$ with $A_{-1} = 0$.

This looks an awful lot like integration by parts! See more in Math 132.

Theorem 4.1. *Let A_n be bounded and b_n monotonically decreasing to 0 then $\sum a_n b_n$ is convergent.*

proof idea. Use Cauchy criterion and sum by parts allows you to bound. □

Example 4.2. $a_n = (-1)^{n+1}$ implies $\sum (-1)^{n+1} b_n$