Lecture 1: August 30, 2017–Motivating Real, sets, relationships, beginning to build our number systems

Goals for the class

My goal is to welcome everyone to mathematics. As a professor, I hold the fundamental belief that everyone in the class is fully capable of engaging and mastering the material. My goal is to meet everyone at least halfway in the learning process. Our classroom should be an inclusive space, where ideas, questions, and misconceptions can be discussed with respect. There is usually more than one way to solve a problem and we will all be richer if we can be open to multiple paths to knowledge. I look forward to getting to know you all, as individuals and as a learning community.

Mathematical goals for the class

My goal is for everyone to

- Learn the content and techniques of real analysis, so that you can creatively solve problems you have never seen before.
- Learn to read and write rigorous proofs, so that you can convincingly defend your reasoning.
- Learn good mathematical writing skills and style, so that you can communicate your ideas effectively and develop your own voice in mathematics.

A Note for students (I borrowed this from another real analysis book, but it has good advice to remind ourselves)

Don't be dismayed if you run into material that doesn't make sense, for whatever reason. It happens to all of us. Just tentatively accept the result as true, set it aside as something to return to, and forge ahead. Also, don't forget to stop me (either during class, at the end of class, or during office hours) if some notation or terminology is puzzling.

Logistics

HW 0 is due Friday at noon outside my door. Part A goes in the Part A folder. Part B in Part B. Information for the course can be found on Sakai. Most logistics questions can be answered by reading the syllabus. Gruders will be contacting you soon with their tutoring hours.

1 What is real analysis?

In real analysis, our proofs will be rigorous and for many of us it is non-intuitive. This rigor is necessary even when an answer may look 'obvious' to us. If we can't articulate exactly why something is true, then we can not use it.

Question 1.1. What is a number? Without thinking too hard, spend 1-2 minutes writing down what you think the definition of a number is. Don't analyze the question, just write what you think.

2 Sets and relations

Definition 2.1. A *set* is a collection of objects.

Example 2.2. Let $S = \{pikachu, charizard, vaporeon, \{mew, arcticuno\}\}$. How many objects are in this set?

Notation:

- $x \in S$ ____
- $x \notin S$ ____
- $\emptyset$ ____
- $A \subset B$ ____

Definition 2.3. If $A \subset B$ but $B \not\subset A$ then A is a proper subset of B.
If $A \subset B$ and $B \subset A$ then $A = B$.

Notation:

- $A \cup B = \{x : x \in A \_\_\_\_ x \in B\}$ ____
- $A \cap B = \{x : x \in A \_\_\_\_ x \in B\}$ ____
- $A^c = \{x \in U : x \notin A\}$ ____
- $A \setminus B = \{x \in A : x \notin B\}$. ____
- $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$. ____

Proposition 2.4. *Suppose A, B, C are subsets of S then*

- $(A \cup B) \cap C =$
- $(A \cap B) \cup C =$
- $(A^c \cap B^c) =$
- $(A^c \cup B^c) =$

Can you prove these?

2.1 Relations

Definition 2.5. A (binary) **relation** R is a subset of $A \times B$. If $(a, b) \in R$ then we write aRb. (The pair (a, b) are in the subset R).

Example 2.6.

- E is “an evolution of” is a relation on $P \times P$ (Pokemon $\times$ Pokemon).
- L ‘Likes’ is a relationship on $P \times C$ (People $\times$ cats).
- $>$ ‘greater than’ is a relation on $\mathbb{Z} \times \mathbb{Z}$.
- $\sim$ ‘the same number’ is a relation on $\mathbb{Q} \times \mathbb{Q}$.

Definition 2.7. An **equivalence relation** on a set S is a relation $\sim$ on $S \times S$ such that

1. $a \sim a$
2. $a \sim b$ implies $b \sim a$
3. if $a \sim b$ and $b \sim c$ then $a \sim c$.

We often denote this relation as: ______

Example 2.8. Question: Which of the above is an equivalence relation? Why or why not? Is there a way we can fix some of these so they may be equivalence relations?

Definition 2.9. A function F from A to B can be written as $F : A \to B$. It is a relation F such that if aFb and aFb' then $b = b'$. This is a very weird way of writing $F(a) = b$.

2.2 Number Systems

We wish to build the real numbers on solid foundation. So we can't always take advantage of things we've always 'known.' For example, I don't actually know that $(a^b)^c$ is a^{bc}. I can prove it for integers using the fact that exponentials with integers means repetitive multiplication (this is its definition). But what does it mean if b, c are rationals or real numbers? So let's carefully define our number systems:

We need to start somewhere, otherwise we'll never start the class. So let us begin at the beginning. Assume we have the integers $\mathbb{Z} =$

and it has the arithmetic we know and love($+$, $\times$ are commutative,associative, has identity, and all that good stuff) and it has an *order*, by this we mean $-3 < -2 < ...$

We want to build the rational numbers formally. Recall from Calculus we defined $\mathbb{Q}$ to be

$\mathbb{Q} =$

This is almost what we need, but let's be more specific.

We want to extend the arithmetic of $\mathbb{Z}$ to $\mathbb{Q}$ so that its operations are "well-defined." What do we mean by well defined? Let's see:

2.2.1 Arithmetic on $\mathbb{Q}$

Let's view $\mathbb{Z}$ as a subset of $\mathbb{Q}$. How?

$n \mapsto$

Addition could be defined as

$$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$$

Is this a good definition of addition? Try an example and see. Remember a single rational number represents infinitely many rational numbers in its equivalence relation.

So addition of two elements in $\mathbb{Q}$ can not change if we choose a different representation of the element from its equivalence class. This is what we mean by well defined.