Lecture 20–Nov 13; Limit definition and Continuity

Learning Goals

- Recognize the need for a formal definition of continuity.
- Define what the limit of a function means and be able to say how it relates to sequences.
- Define continuity at a point.
- State and prove properties of continuity and how it relates to limits.
- State the topological definiton of continuity.

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1 Continuity motivation

Activity to motivate section; Consider the Fourier series of $f(x) = \frac{1}{2} - |x - \frac{1}{2}|$ which is

$$f(x) = \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin((2k+1)\pi x).$$

1. Graph

$$f(x) = \frac{4}{\pi^2} \sum_{k=0}^{5} \frac{(-1)^k}{(2k+1)^2} \sin((2k+1)\pi x).$$

2. What observation can be made as you let $n \mapsto \infty$ in the partial sum. As you increase n how does the graph change?

Where is the graph converging to as $n \mapsto \infty$? Draw it below.

3. Take the derivative of the original function and graph it.

4. Take the derivative term by term in the series and graph it. What do you notice?

5. In the series you created in 4: Is each term in the sequence of partial sums continuous? Do you think the series is continuous?

6. What do you expect the graph of the derivative to look like based on 4.? Use your knowledge of derivatives to make an educated guess.

7. Differentiate the series term by term and graph it. What did you get? Does this behaviour match what you guessed should happen in the previous question?

Conclusion: An infinite sum of continuous functions is not continuous! Power series are special and behave well, but Fourier series have something going on that makes them badly(awesomely) behaved. This is another example where the reliance on intuition can be perilous. Rigour is needed to either justify or dismiss our intuition. This is what 19th century mathematicians spent a good deal of time figuring out!

Turns out there are different concepts of continuity and we should carefully sort them all out. First we need to know more about limits.

2 Limits for functions

Recall, we define $\lim_{n\mapsto\infty} x_n = a$. We would like a notion for what it means for $\lim_{x\mapsto p} f(x) = q$.

Let X and Y be metric spaces and $f : X \to Y$ be a function.

We can visualize f either by graphing or by mapping:

Definition 2.1. Let X, Y be metric spaces, $E \subset X$, p is a limit point of E, and $f : E \to Y$. We say $f(x) \mapsto q$ as $x \mapsto p$ or

$$\lim_{x\mapsto p} f(x) = q$$

if there exists $q \in Y$ such that for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in E$ if $0 < d_X(x,p) < \delta$ then $d_Y(f(x), q) < \epsilon$.

Remark 2.2. $f : \mathbb{R} \to \mathbb{R}$ has notion of left and right hand limit where domain E is restricted to one side.

Theorem 2.3. $\lim_{x \mapsto p} f(x) = q$ *if and only if for all sequence* $\{p_n\}$ *in* E *such that* $p_n \neq p$ *and* $p_n \mapsto p$ *we have* $f(p_n) \mapsto q$.

Proof. Forward direction: Suppose we are given a sequence $p_n \neq p$ and $p_n \mapsto p$. We want to show $f(p_n) \mapsto q$.

Backward direction: By contradiction. Suppose

$$\lim_{x \mapsto p} f(x) \neq q.$$

We will find a bounded sequence $p_n \mapsto p$ but $f(p_n) \not\mapsto q$.

□

Corollary 2.4. *A consequence of the previous theorem is*

1.
$$\lim_{x \mapsto p} f(x)$$
is unique.

2. *For $f : \mathbb{R} \to \mathbb{R}$ the sum of the limits is the limit of the sums and the product of the limits is the limits of the product.*

See Rudin for other useful properties of limits.

Proof idea. Part a: Rudin 3.2b and previous theorem.

Part b: Rudin 3.3 and previous theorem. □

3 Continuous functions

Definition 3.1. Let X and Y be metric spaces. Let $p \in E \subset X$, and $f : E \to Y$. We say f is **continuous at** p if for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in E$ if $d(x, p) < \delta$ then $d(f(x), f(p)) < \epsilon$.

We say 'f is continuous on E' to mean f is continuous on all points of E.

Note: No restriction on x, x can be p. Notice this definition is different from that of the limit, because we require $f(x)$ be near $f(p)$, not just q.

Intuition for continuity:

So our task is given ϵ find a δ which may depend on ϵ and p.

Example 3.2. Examples of continuity.

Continuous functions preserve limits.

Theorem 3.3. *If p is a limit point of E then f is continuous at $p \in E$ if and only if*

$$\lim_{x \mapsto p} f(x) = f(p).$$

Hence f is continuous on E if and only if for all convergent sequences $\{x_n\}$ in E

$$\lim_{n \mapsto \infty} f(x_n) = f(\lim_{n \mapsto \infty} x_n).$$

Proof idea. Use limit definitions.

□

Corollary 3.4.

1. *If $Y = \mathbb{R}$ then finite sums and products of continuous functions are continuous.*
2. *If $Y = \mathbb{R}^k$, $f = \langle f_i \rangle$ is continuous iff all components f_i are continuous.*

Recall;

Definition 3.5. $f^{-1}(U) = \{x : f(x) \in U\}$ is the inverse image of U.

Theorem 3.6. *$f : X \to Y$ is continuous iff for all open sets U in Y $f^{-1}(U)$ is open in X.*

This is the topological definition of continuity.

Proof. NEXT TIME! □