Lecture 21–Nov 15; Consequences of Continuity

Learning Goals

- Prove that the topological definition of continuity is equivalent to our defintion of continuity.
- Use the topological definition of continuity to prove many consequences of continuity.
- Prove continuous images of compact and connected spaces are still compact and connected.
- Understand that the Intermediate Value Theorem is a consequence of continuity and connectedness.

HW 9 due Friday at noon. Hw 10 due Wednesday in class.

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1 Topological definition of Continuity

Recall:

Definition 1.1. $f^{-1}(U) = \{x : f(x) \in U\}$ is the inverse image of U.

Theorem 1.2. *$f : X \to Y$ is continuous iff for all open sets U in Y $f^{-1}(U)$ is open in X.*

This is the topological definition of continuity.

Proof. Forward direction. We assume f is continuous. Pick $p \in f^{-1}(U)$. We want to show that p is an interior point of $f^{-1}(U)$.

Backward direction. Fix $p \in X$ and $\epsilon > 0$. We must find δ such that $d(p, x) < \delta$ implies $d(f(p), f(x)) < \epsilon$.

□

Corollary 1.3. *$f : X \to Y$ is continuous iff $f^{-1}(C)$ is closed in X for every closed set C in Y.*

Proof idea. Recall a set is closed iff its complement is open and

$$f^{-1}(E^c) = (f^{-1}(E))^c$$

for all $E \subset Y$.

□

Corollary 1.4. *Consider $f : X \to Y$ and $g : Y \to Z$, where f and g are continuous functions. Then $g \circ f$ is continuous.*

Proof. Pick U open in Z.

□

2 Continuity consequences

Theorem 2.1. *If $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact. So continuity preserves compactness.*

Proof. Consider an open cover of $f(X)$; $\{U_\alpha\}$.

□

Corollary 2.2. *If $f : X \to \mathbb{R}^k$ and X is compact then $f(X)$ is closed and bounded. Hence f is bounded.*

Corollary 2.3. *A continuous function $f : X \to \mathbb{R}$ with X compact must achieve a max, $M = \sup_{p\in X} f(p)$ and a min, $m = \inf_{p\in X} f(p)$. (There exists a q and r such that $f(q) = M$ and $f(r) = m$.)*

Theorem 2.4. *Let $f : X \to Y$ be continuous and E connected in X, then $f(E)$ is connected. So continuity preserves connected sets.*

proof idea. If $f(E)$ is not connected then there exists an A and B, nonempty, such that $\bar{A} \cap B = A \cap \bar{B} = \emptyset$ and $A \cup B = f(E)$.

□

Corollary 2.5 (Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous and $f(a) < c < f(b)$ then there exists an $x \in (a, b)$ such that $f(x) = c$.*