Lecture 22–Nov 20; Uniform Continuity

Learning Goals

- State the definition of uniform continuity.
- Recognize which continuous functions are uniformly continuous or not.
- Be able to prove a space is uniformly continuous using the defintion.
- State and prove Lebesgue Covering Lemma.
- Define and recognize the two types of discontinuities.

Recall: $f : X \to Y$ is continuous if it fulfills the epsilon-delta definition: For all $p \in X$ and for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in X$ if $d(x, p) < \delta$ then $d(f(x), f(p)) < \epsilon$. So we may need a different δ-ball depending on p.

iff

For all convergent sequence $\{x_n\}$ the $\lim_{n \mapsto \infty} f(x_n) = f(\lim_{n \mapsto \infty} x_n)$. "continuous functions preserve limts"

iff

for all open U in Y $f^{-1}(U)$ is open in X. "inverse images of open sets are open"

iff

for all closed C in Y $f^{-1}(C)$ is closed in X. "inverse images of closed sets are closed."

Consequences so far

- continuous functions preserve compactness.
- continuous functions preserve connectedness. This gives us IVT for the real numbers!

1 Uniform Continuity

Let's work with a new type of continuity.

Definition 1.1. We say $f : X \to Y$ is **uniformly continuous** if for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all x and p in X if $d_X(x, p) < \delta$ then $d(f(x), f(p)) < \epsilon$.

Question: How does this definition differ from continuous at a point?

Example 1.2. Give an example of a continuous function that is not uniformly continuous.

Note: Uniform continuity implies continuity.

Theorem 1.3. *$f : X \to Y$ is continuous and X is compact then f is uniformly continuous on X.*

Proof. In Rudin; He shows that open covers of δ-balls have finite subcovers since X is compact. We will take another approach.

Fix $\epsilon > 0$ our goal is to find a δ *for all* p.

□

We need the following lemma to find the δ.

Theorem 1.4 (Lebesgue Covering Lemma)**.** *If $\{U_\alpha\}$ is an open cover of a compact metric space then there exists a $\delta > 0$ such that for all $x \in X$, $B(x, \delta) \subset U_\alpha$.*

Proof Sketch. X is compact then there exists a finite subcover $\{U_{\alpha_i}\}$.

□

2 Discontinuities

Definition 2.1. We can define

$$f(x^-) = \lim_{t \mapsto x^-} f(t)$$

to be the **left hand limit** and

$$f(x^+) = \lim_{t \mapsto x^+} f(t)$$

to be the **right hand limit**. We say the limit as $t \mapsto x$ exists when $f(x^-) = f(x^+)$.

Example 2.2. If x is a point in the domain of definition of the function f at which g is not continuous we say that f has *discontinuity at x*. If f is defined on a segment or an interval it is customary to classify discontinuity into two types.

- If f is discontinuous as x and $f(x^-)$ and $f(x^+)$ exists we say f has a **discontinuity of the first kind**, or a **simple discontinuity** at x.

- Else we say f has a discontinuity of the **second kind.**

- Dirchilet function

- HW problem: