Lecture 23–November 22; Differentiability

Learning Goals

- Define the limit definition of differentiabilty.
- Recognize the relationship between continuity and differentiability and learn the notation for functions with continuous derivatives.
- Know the rules of differentiability and be able to use them.
- State the Mean Value Theorem and be able to use it to prove other statements. Recognize the importance of the MVT.
- Practice better proof writing by reading other people's proofs.

Hw 10 due today. Hw 11 due next Friday at 12pm, there are 8 problems.

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1 Differentiability

Recall: What is the rate of change of f?

Definition 1.1. A function $f : [a, b] \to \mathbb{R}$ is **differentiable** at $x \in [a, b]$ if the following limit exists:

$$\lim_{t \mapsto x} \frac{f(t) - f(x)}{t - x}$$

where $t \neq x$ and $t \in (a, b)$.

We say "the derivative of f at x," or the "instantaneous rate of change."

Notation:

Question: Does continuity imply differentiablility?

Question: Does differentiability imply continuity?
Guesses:

Theorem 1.2. *Let f be defined on $[a, b]$. If f is differentiable at a point $x \in [a, b]$, then f is continuous at x.*

Proof. If $t \mapsto x$ then

$\square$

Question: If f is differentiable on $[a, b]$ is f' continuous on $[a, b]$?
Guesses:

Levels of differentiability. We say a function is:

- $\mathcal{C}^0$ if the function is continuous.

- $\mathcal{C}^1$ if it is differentiable and the derivative is continuous.

- $\mathcal{C}^k$ if the function's kth derivative exists and the kth derivative is continuous.

- $\mathcal{C}^\infty$ if all the derivatives exists. We call $\mathcal{C}^\infty$ functions "smooth."

- $\mathcal{C}^\omega$ if the function has a power series expansion at each point. We refer to these functions as "analytic." (See more of these in complex analysis).

Remark 1.3. We defined the derivatives to be a limit, so sums, products and quotient rules follow.

Example 1.4.

- $(f \pm g)' = f' \pm g'$.

- $(fg)' = f'g + g'f$.

Arguably the most important property about derivatives:

Theorem 1.5. *Mean Value Theorem. If f is continuous on $[a, b]$ and f is differentiable on (a, b) then there exists a $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.*

A (rough) bit of history:

- A special case was described by Parameshvara in the 13-1400s.
- A restricted form was proved by Rolle in 1691, not using infinitesimals.
- Cauchy proves MVT (with a not quite correct proof) in 1823.
- Proved by Bonnet in 1868.

This theorem awesome and important because it is the only theorem we have that relates the values of f to the values of f' without appealing to limits.

Example 1.6. An application:
If $f'(x) > 0$ for all $x \in (a, b)$ then f is strictly increasing in (a, b).

Proof. Let $s < t$ in (a, b) then

□

Question: Why do we need to assume differentiable?

Note there is more to differentiability that what we will cover in lecture. Please see Rudin page 105 for proof of the chain rule and page 109 for L'Hôpital's rule.

2 Proof writing activity

You chose a theorem from this class (either one from Rudin or from your homework) that you could do the following for: Write a clear, complete, and valid proof of your chosen theorem. Then you wrote an invalid 'proof' of the theorem, making the invalid proof as convincing as possible. And you provided an example that illustrates what your theorem says.

- Now we will work in teams of two on the "proofs" you are given and determine which is the valid proof and which is the invalid proof and why. Provide an example or explanation of why the invalid proof does not work.
- Read the correct proof and critique the proof. Was the proof clear and concise? Can you imagine the you from the beginning of the semester being able to follow the logic and reasoning in the proof or is more explanation or citation needed?
- Did the proof follow the guideline for good mathematical writing, https://www.math.hmc.edu/~su/math131/good-math-writing.pdf?