Lecture 24–Nov 27; Taylor's Thereom and Uniform Convergence

Learning Goals

- State and prove MVT and related Theorems.
- State and prove Taylor's Theorem and understand its relationship with MVT.
- Define pointwise convergence of a sequence of functions. Be able to find pointwise convergence for nice functions.
- Define uniform convergence of a sequence of functions and see how it's needed to imply continuity.

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1 Differentiability continued

Recall:

Theorem 1.1. *Mean Value Theorem. If f is continuous on $[a, b]$ and f is differentiable on (a, b) then there exists a $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.*

Question: Why do we need to assume differentiable?

Note there is more to differentiability that what we will cover in lecture. Please see Rudin pages 108-113. Your homework will point you to theorems to think about that were not discussed in class.

Theorem 1.2. *Rolle's Theorem, a special case of MVT. If $h : [a, b] \to \mathbb{R}$ has a local max at $c \in (a, b)$ and $h'(c)$ exists then $h'(c) = 0$.*

Proof Idea. Look at the signs of the slopes of the secant lines on the left and right .

□

Theorem 1.3. *Generalized MVT (Cauchy). If f and g is continuous on $[a, b]$ and differentiable on (a, b) then there exists a $c \in (a, b)$ such that*

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

Proof. Proof Idea. Let $f(t) =$ the position of the knive C at time t on the x-axis. Let $g(t) =$ the position of the knife D at time t on the y-axis.

□

2 Taylor's Theorem

Note in MVT:

$$f(b) = f(a) + \underbrace{f'(c)(b - a)}_{\text{error term}}$$

for some $c \in (a, b)$. The error term is not precisely known because it is hard to find the c.

This suggests that $f(b) = f(a) + f'(a)(b - a)+$error. In fact the error term is $f''(c)(b - a)^2/2!$.

Definition 2.1. In general, $P_n(x) = f(a) + f'(a)(x-a) + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n$ is the n**th Taylor Polynomial.**

Remark 2.2.

- This polynomial has the same value and n derivatives as f at a.
- This is the "best" polynomial approximation of degree n.
- This leads to the question: How good is this approximation?

Theorem 2.3. *Taylor's Theorem*
If $f^{(n-1)}$ is continuous on $[a,b]$ and $f^{(n)}$ exists on (a,b) then P_{n-1} approximates f for $x \in (a,b)$:

$$f(x) = P_{n-1}(x) + \underbrace{\frac{f^{(n)}(c)}{n!}(x-a)^n}_{error}$$

for some $c \in (a,x)$.

Note:

Proof. Proof Sketch. If $n = 1$ then

□

Proof. Alternative Proof.

- Let $P(a,x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k$.
- Define K by $f(b) = f(a) + \frac{K}{n!}(b-a)^n$.
- We want to show that $K = f^{(n)}(c)$ for some $c \in (a,b)$.

□

2.1 Ch 7; Sequences of Functions

We know what it means for a sequence of numbers to converge. But functions change depending on their input. That leads us to the question:

Question: What does it mean to say $f_n(x)$ converges?

Definition 2.4. A natural way to define convergence of a sequence of functions is *pointwise*: Fix an x. Does $\{f_n(x)\}$ converge as a sequence of points? If $\{f_n(x)\}$ converges to $f(x)$ for all x then the *pointwise limit* is $f(x) = \lim_{n\mapsto\infty} f_n(x)$.

Example 2.5.

- $f_n(x) = x/n$ on $\mathbb{R}$.

- $f_n(x) = x^n$ on $[0, 1]$.

- What about weirder examples?

Definition 2.6. For bounded : $E \to \mathbb{R}$ define

$$||f|| = \sup_{x \in E} |f(x)|.$$

We call this the **sup norm**.

Definition 2.7. We say $f_n \mapsto f$ "f_n **converges uniformly** to f" if for all $\epsilon > 0$ there exists N such that for all $n \geq N$ implies $||f_n - f|| < \epsilon$.

The intuition here is that this is the "ribbon convergence distance."

Example 2.8. Returning to our previous examples:

- Our first example does not converge uniformly. Why?

  What could we do to the domain so that it would converge uniformly?

- Second example:

Theorem 2.9. *If f_n is continuous and f_n converges uniformly to f then f is continuous.*

Proof. We want to use the bound:

$$|f(x) - f(y)| \leq \underbrace{|f(x) - f_n(x)|}_{(1)} + \underbrace{|f_n(x) - f_n(y)|}_{(2)} + \underbrace{|f_n(y) - f(y)|}_{(3)}.$$

□