Lecture 25–Nov 29; Convergence of functions

Learning Goals

- Be able to define two forms of sequence of functions convergence.
- Recognize the difference between the two types of convergence.
- Be able to recognize if a sequence of function converges pointwise or uniformly.
- Explore what properties of functions are preserved by these limit defintions of convergence. Recognize we need uniform convergence if we want to preserve continuity.

For now we restrict to complex valued functions. Though many of our results extend to vector valued functions and some generalize to general metric spaces.

1 Ch 7; sequences of functions

What does it mean to say $f_n(x)$ converges?

A natural way to define convergence of a sequence of functions is *pointwise*. For a sequence of functions;

Definition 1.1. Suppose $\{f_n\}$ is a sequence of functions defined on a set E. If $\{f_n(x)\}$ converges for every x then we say f_n converges *pointwise* to the limit $f(x) = \lim_{n\mapsto\infty} f_n(x)$, $x \in E$.

Further

Definition 1.2. If $\sum f_n(x)$ converges for every $x \in E$ and we define

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

for $x \in E$, the function f is called the *sum* of the series $\sum f_n$.

Example 1.3.

- $f_n(x) = \frac{x}{n}$ on $\mathbb{R}$.

Note

- Question: If f_n converges pointwise to f and each f_n is continuous for all $n \in \mathbb{N}$, is f continuous?

  Guesses?

  Consider$f_n(x) = x^n$ on $[0, 1]$.

  This demonstrates

- Recall the Fourier series for $f(x) = \frac{1}{2} - |x - \frac{1}{2}|$ is

$$f(x) = \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin((2k+1)\pi x).$$

Let $f_n(x)$ be

$$f_n(x) = \frac{4}{\pi^2} \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)^2} \sin((2k+1)\pi x).$$

- What about weirder examples?

Remark 1.4. The main question we have for sequences of functions is we want to see what important properties are preserved under the limit operation: differentiability, integrability? We just saw continuity is not preserved for pointwise convergence.

But

Recall:

Definition 1.5. For bounded : $E \to \mathbb{R}$ define

$$||f|| = \sup_{x \in E} |f(x)|.$$

We call this the **sup norm**.

Definition 1.6. We say $f_n \mapsto f$ "f_n **converges uniformly** to f" if for all $\epsilon > 0$ there exists N such that for all $n \geq N$ implies $||f_n - f|| < \epsilon$.

The intuition here is

Example 1.7. Returning to our previous examples:

- $f_n(x) = \frac{x}{n}$

  What could we do to the domain so that it would converge uniformly?

- $f_n(x) = x^n$

Theorem 1.8. *If f_n is continuous and f_n converges uniformly to f then f is continuous.*

Proof. We want to use the bound:
$|f(x) - f(y)|$

□

So strengthening our convergence definition does preserve continuity.

Question, is the Fourier series or Power series uniformly convergent?

Let's return to other properties we want preserved. Is differentiability preserved? That is, If f_n converges pointwise (or uniformly) to f, and each f_n is differentiable, will f be differentiable?
Guesses:

Theorem 1.9 (Weierstrauss 1872.). *The function*

$$f = \sum b^n \cos(a^n \pi x)$$

with $0 < b < 1$, $a \in 2\mathbb{Z}+1$ *and* $ab > 1 + 3\pi/2$ *is continuous everywhere, but differentiable nowhere.*

Proof idea. We want to start with an arbitrary point $x_0 \in \mathbb{R}$. Then we will examine the left and right hand limits of

$$\frac{f(y) - f(x_0)}{y - x_0}$$

as $y \mapsto x_0$.

For details of this proof see: http://www.ohiouniversityfaculty.com/mohlenka/20132/4993-4994/20132McLaughlin_report.pdf The algebra is messy and the author begins with a clever choice of y_m and z_m and does not share on the intuition of where those values come from. But for the curious among you, none of the math used is beyond the scope of this class. □