Lecture 2–September 4: Building the rational numbers

Learning Goals

- We create a solid foundation for real numbers. Students will learn how to extend operations on integers to the rational numbers that are well defined.
- Be able to define a field and ordered field.
- Students will be able to prove that the rational numbers are dense.
- Students will be able to show that the rational number line has holes.

1 Number systems

We wish to build the real numbers on solid foundation. So we can't always take advantage of things we've always 'known.' For example, I don't actually know that $(a^b)^c$ is a^{bc}. I can prove it for integers using the fact that exponentials with integers means repetitive multiplication (this is its definition). But what does it mean if b, c are rationals or real numbers? So let's carefully define our number systems:

We need to start somewhere, otherwise we'll never start the class. So let's assume we have the integers

$\mathbb{Z} =$ ______________________

Definition 1.1. .

- Let S be a set. An **order** on S is a relation $<$ with the following properties:
  1. If $x \in S$ and $y \in S$ then one and ONLY one of the following holds: $x < y, y < x, x = y$.

     ____
  2. If $x, y, z \in S$ if $x < y$ and $y < z$ then $x < z$. ______________________

  We read $x < y$ as ______________________. We often say $y > x$ in place of $x < y$. $x \leq y$ indicates $x < y$ OR $x = y$.
- An **ordered set** is a set S in which an order is defined, and we write the ordered set as $(S, <)$.

Example 1.2. $\mathbb{Z}$ is an ordered set with order______________________. Show $<$ is an order on $\mathbb{Z}$.

We want to build the rational numbers formally. Recall from Calculus we defined $\mathbb{Q}$ to be

$\mathbb{Q} =$ ________________________

This is almost what we need, but let's be clearer.
$\frac{p}{q}$ is an equivalence class defined by the equivalence relations $\sim$ on $\mathbb{Z} \times \{\mathbb{Z} \setminus 0\}$.

We define the equivalence to be $(p, q) \sim (m, n)$ iff ________________________.

Question: Is this an equivalence relation?
Reflexive:

Symmetric:

1.1 Arithmetic on $\mathbb{Q}$

We want to extend the arithmetic of $\mathbb{Z}$ to $\mathbb{Q}$ so that its operations are "well-defined." What do we mean by well defined?

Let's view $\mathbb{Z} \subset \mathbb{Q}$. How?

$$n \mapsto \underline{\hspace{4cm}}.$$

Let's try to extend addition. One defintion we may use is the following:

$$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$$

Is this a good definition of addition? Try an example and see. Remember a single rational number represents infinitely many rational numbers in its equivalence relation.

We can also try

$$\frac{a}{b} + \frac{c}{d} = \frac{0}{1}$$

Is this addition well defined?

This definition is BORING! Here different representation give the same output. But it's a little too same for us.

We have to find a way of adding to elements that is well defined.
The Goldie Locks definition (it's just right):

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

This extends addition on $\mathbb{Z}$ to addition on $\mathbb{Q}$. Is this addition well defined? We need to check if

________________________ then ________________________.

How would you define multiplication and what would you need to check to show well defined? You will show these on your homework.

Cool fact! The order on $\mathbb{Z}$ can be extended to $\mathbb{Q}$.

Example 1.3. Our above example says for $\mathbb{Z}$, $m < n$ if $n - m \in \{1, 2, 3, ...\}$.

In $\mathbb{Q}$ we will say $\frac{a}{b} < \frac{c}{d}$ if ________________________..

Where $\frac{m}{n}$ is positive if either both m, n are positive or both are negative.
Question: Is this well defined? Check:

$<$ is an order:

$<$ is well defined:

2 Fields and ordered fields

Now with $\mathbb{Q}$ equipped with this addition, multiplication and order we can see that $\mathbb{Q}$ is a **field**. ($\mathbb{Z}$ is a **ring**).

Definition 2.1. A **field** is a set F with two operations, called addition and multiplication, which satisfy the field axioms: A set $(F, +, \times)$ with operations is a **field** if it satisfies the following axioms:

1. closed. ____________________

2. commutative. ____________________

3. associative. ____________________

4. multiplicative identity and an additive identity. ____________________

5. Multiplicative and additive inverses exist. ____________________

6. the operations play nice: distributive law holds. ____________________

Now check if $\mathbb{Q}$ is a field:

When $\mathbb{Q}$ is equipped with the order extended from $\mathbb{Z}$ it becomes an *ordered field.*

Definition 2.2. An **ordered field** is a field F with order $<$ which is preserved by operations:

i If $x, y, z \in F$ and $y < z$ then $x + y < x + z$.

ii If $x, y, z \in F$, $y < z$, and $x \neq 0$ then $xy < xz$.

ii' If $x, y \in F$ with $x > 0$ and $y > 0$ then $xy > 0$.

So all this says $\mathbb{Q}$ is great (number theorists love it). For example:

Example 2.3. $\mathbb{Q}$ solves equations $\mathbb{Z}$ could not. Consider $3x + 5 = 0$.

There is a lot to $\mathbb{Q}$:

Theorem 2.4. *Between any* 2 *distinct rational numbers there is a rational number.*

Proof. Let ________________________.

□

But! $\mathbb{Q}$ is still not good enough.

Theorem 2.5. $x^2 = 2$ *has no solutions in* $\mathbb{Q}$.

Proof. We will prove this using contradiction.

□

So there are holes in the number line, we can't see them, but they are there.