Lecture 3–September 6–Construction of $\mathbb{R}$

Learning Goals

- Define least upper bound property.
- Define Dedekind cuts and use them to define $\mathbb{R}$
- Build arithmetic on $\mathbb{R}$.

FRIDAY September 8: HW 1 DUE at **12pm**

Cool fact, we might find it useful:

Theorem 0.1. *$\mathbb{Q}$ has the archimedean property. That is if $x, y \in \mathbb{Q}$ then there exists $n \in \mathbb{Z}^+$ such that $nx > y$.*

Question: How can we extend $\mathbb{Q}$ such that it has no holes?
We begin by defining some new concepts

Definition 0.2. Say $E \subset S$ is ordered. If there exists $\beta \in S$ such that for all $x \in E$ we have $x \leq \beta$ then

we say E is ________________ and we call β the ________________.
Notice $\beta \in S$ but is not necessarily in E.
We can define the lower bound similarly. What would the definition be?

Example 0.3. $\mathbb{Q}_- =$________________ has upper bounds ________________

Definition 0.4. To say α is a ________________________ of E means

1. α is an upper bound of E and
2. If $\gamma < \alpha$ then γ is not an upper bound of E.

We write $\alpha = \sup E$ or $\alpha = \text{lub}E$.

Example 0.5. Let $S = \mathbb{Q}$. Then

- $\sup\{1/2, 1, 2\} =$ ________________
- $\sup\{\mathbb{Q}_-\} = 0$________________
- $\sup\{r : r < 2\} =$ ________________

  If $x \in \{r : r < 2\}$ then ________________

- $\sup\{r : r^2 < 2\}$______________________________________

The last example demonstrates that there are sets in $\mathbb{Q}$ with no least upper bound. This is a big problem if we want to do calculus since it assumes everything is continuous.

Definition 0.6. A set S has the **least upper bound property** if every nonempty subset of S that has an upper bound has a least upper bound in S.

Example 0.7. Consider $\mathbb{Q}$.

1 Construct the reals

We wish to construct $\mathbb{R}$ which will "fill in the holes" in $\mathbb{Q}$.

The method we will use is **Dedekind cuts**.

1.1 Step 1: define cuts

Definition 1.1. A **cut** α is a subset of $\mathbb{Q}$ such that

1. $\alpha \neq \emptyset$ and $\alpha \neq \mathbb{Q}$______________________

2. If $p \in \alpha$ and $q < p$ with $q \in \mathbb{Q}$ then $q \in \alpha$______________________

3. If __________________________________

Example 1.2. An example of a cut

Definition 1.3. Let $\mathbb{R} = \{\alpha : \alpha \text{ is a cut}\}$.

Now we have a set. We want it to have an arithmetic and order extended from $\mathbb{Q}$ (Recall, this is what we had to do for $\mathbb{Q}$ when we only had $\mathbb{Z}$).

Let α, β, γ be cuts.

1.2 Step 2: define the ordering of $\mathbb{R}$

For cuts α and β we define $\alpha < \beta$ to mean $\alpha \subsetneq \beta$.

This gives us an order on $\mathbb{R}$. Please check:

- If $\alpha < \beta$ and $\beta < \gamma$ is $\alpha < \gamma$?

- Trichotomy? To show that at least one holds, assume that the first two fail.

1.3 Step 3: Show the ordered set $\mathbb{R}$ has the lub property

Theorem 1.4. $\mathbb{R}$ *has lub property.*

The idea is to consider

See more next week.

We will use this property to characterize $\mathbb{R}$ and never again will we use cuts after next Monday.

1.4 Step 4: Addition– $\alpha, \beta \in \mathbb{R}$ then define $\alpha + \beta$ and show field axioms hold

Define addition to be

$$\alpha + \beta = \{r + s : r \in \alpha \text{ and } s \in \beta\}.$$

Does this fulfill the field axioms? Check

- $\alpha + \beta$ is still a cut. We need to check three properties:
  1. $\alpha + \beta$ is nontrivial.

  2. It's closed downward. Pick $p \in \alpha + \beta$

  3. Pick $p \in \alpha + \beta$. ______________________________

- $\alpha + \beta = \beta + \alpha$: We know $r + s = s + r$________________________

- $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$________________________

- Identity is $\mathbb{Q}_- = 0^*$. Please check that this is the identity. If $r \in \alpha$ and $s \in 0^*$ then

  __________________.

  Now suppose $p \in \alpha$ and pick ________________________

- There exists inverses for α. Use $\beta = \{p : \exists r > 0 \text{ st } -p - r \notin \alpha\}$.

1.5 Step 5: Prop 1.14 holds now show $\mathbb{R}$ is an ordered field

Proposition 1.5. *Proposition 1.14 in Rudin*
The axioms of addition imply the following:

1. *If $x + y = x + z$ then $y = z$.*
2. *If $x + y = x$ then $y = 0$.*
3. *if $x + y = 0$ then $y = -x$.*
4. *$-(-x) = x$.*

To show $\mathbb{R}$ is an ordered field we need to check
$x + y < x + z$ if $x, y, z \in \mathbb{R}$ and $y < z$

and check $xy > 0$ if $x, y \in \mathbb{R}$, $x > 0$, $y > 0$.

1.6 Step 6: Multiplication–define multiplication of positive reals

We define $\alpha\beta$ for positive cuts first: $\alpha, \beta > 0^*$. We define

$$\alpha\beta := \{p : p < rs, r \in \alpha \text{ and } s \in \beta, r > 0, s > 0\}.$$

We would need to check that the field axioms hold for this multiplication.

1.7 Step 7: Multiplication–extend the definition and show field axioms hold

$$\alpha\beta = \begin{cases} (-\alpha)(-\beta) & \text{if } \alpha < 0^*, \beta < 0^* \\ -[(-\alpha)\beta] & \text{if } \alpha < 0^*, \beta > 0^* \\ -[\alpha(-\beta)] & \text{if } \alpha > 0^*, \beta < 0^* \end{cases}$$

Steps 1-7 give us that $\mathbb{R}$ is a field with lub property.

1.7.1 Step 8: Associate $\mathbb{Q}$ to $\mathbb{Q}^*$ and show operations behave nicely

- Given $r \in \mathbb{Q}$ let $r^* = \{q \in \mathbb{Q} : q < r\}$. This is the cut corresponding to a rational number. Thus $\mathbb{Q}^* \subset \mathbb{R}$.
- Show
  1. $r^* + s^* = (r + s)^*$
  2. $r^* s^* = (rs)^*$
  3. $r^* < s^*$ iff $r < s$.

1.7.2 Step 9: $\mathbb{Q}$ and $\mathbb{Q}^*$ are isomorphic

$\mathbb{Q}$ and $\mathbb{Q}^*$ are not equal, but they are isomorphic. Which means we can map between these sets and preserve the properties we care about. Namely addition, multiplication, and order. With this identification we can view $\mathbb{Q}$ as a subset of $\mathbb{R}$