Lecture 4–Sept 11: Least upper bound property

Learning Goals

- Prove $\mathbb{R}$ has the least upper bound property.
- Be able to use the least upper bound property to show other interesting properties of $\mathbb{R}$: the archimedean property; $\mathbb{Q}$ is 'dense' in $\mathbb{R}$; general roots exist.

Typos last week:

Theorem 0.1. *$\mathbb{Q}$ has the archimedean property. That is if $x, y \in \mathbb{Q}$* ***and*** *$x > 0$ then there exists $n \in \mathbb{Z}^+$ such that $nx > y$.*

Recall that we defined $\mathbb{R}$ to be

$$\mathbb{R} := \{\alpha : \alpha \text{ is a cut}\}.$$

α is a cut if its nontrivial, closed downward, and has no biggest element. This means:

Example 0.2. $\{r \in \mathbb{Q} : r \leq 2\}$ is ________________________________

We defined arithmetic of cuts and an order. We also found that the identity elements is $0^* = \{r : r < 0\}$.
We saw that $\mathbb{Q}$ is embedded in $\mathbb{R}$ via:

$$r \in \mathbb{Q} \mapsto r^* = \{q \in \mathbb{Q} : q < r\}.$$

We say r^* is a **rational cut**.

Example 0.3. $\frac{1}{2}^*$ ____________________

We see that ____________________

Question: How do we know there is a cut α that is not a rational cut?

We will be able to eventually show that general roots of real numbers exist.
Today we show the following:

Theorem 0.4. *$\mathbb{R}$ is an ordered field that contains $\mathbb{Q}$ as a subfield and it has lub property.*

Proof. Last time we proved that $\mathbb{R}$ has order and arithmetic that extends from $\mathbb{Q}$, and we began to prove that $\mathbb{R}$ is an ordered field. See Rudin pg 18-21 to complete the proof that $\mathbb{R}$ is an ordered field if curious. We saw above that $\mathbb{Q}$ can be embedded in $\mathbb{R}$. . Now we show that $\mathbb{R}$ has the least upper bound property.

We will show that $\mathbb{R}$ has the lub property.

Let A be a nonempty subset of $\mathbb{R}$ and assume $\beta \in \mathbb{R}$ is an upper bound of A.

Define γ to be ____________________

We will show ____________________

We first show γ is a cut:

Now we show γ is the lub:

□

Remark 0.5. The **least upper bound property of** $\mathbb{R}$ is the 'completeness axiom' aka there are no holes in $\mathbb{R}$.

Now we will think of $\mathbb{R}$ as an ordered field containing $\mathbb{Q}$ with the least upper bound property. As of right now, we are done with Dedekind cuts. We will never use these to prove anything again, please don't use it in your proofs unless explicitly asked to use cuts.

1 Consequences of LUB property

Theorem 1.1. *$\mathbb{R}$ has the archimedean property.*

That is if ____

Proof. Let $A = \{nx : n \in \mathbb{N}\}$.

$\square$

Theorem 1.2. *For any $x, y \in \mathbb{R}$ with $x < y$ there exists $q \in \mathbb{Q}$ such that $x < q < y$.*

Proof. Use Archimedean property to prove that $\mathbb{Q}$ is 'dense' in $\mathbb{R}$.

Note $y - x > 0$ so the $\overbrace{\text{archimedean property}}^{\rule{6em}{0.4pt}}$ implies there exists $n \in \mathbb{N}$ such that $n(y - x) > 1$. Choose first $m \in \mathbb{Z}$ such that

$\square$

Theorem 1.3. *General roots exist*

For every real number $x > 0$ and every integer $n > 0$ there is one and only one positive real y such that $y^n = x$. We write y as $x^{1/n}$ or $\sqrt[n]{x}$.

Proof. Sketch. This is an existence and uniqueness theorem so we much show two things.

Uniqueness:

Existence:

∎

A consequence of Theorem 1.3:

Corollary 1.4. *If $a, b \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$ then $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$.*

Next time: Other number systems!