Lecture 5–Sept 13: Other number systems

Learning Goals

- Define other number systems that are interesting to look at and may appear again in 131: $\overline{\mathbb{R}}, \mathbb{R}^k, \mathbb{C}, \mathbb{C}^k$.
- Define distance on $\mathbb{R}^k$ and $\mathbb{C}^k$.
- State and prove Cauchy-Schwarz inequality for $\mathbb{C}^k$.

FRIDAY September 15: HW 2 DUE at **12pm**

Last time: We discussed the least upper bound property $\sup E = \operatorname{lub} E$.
Properties of least upper bounds:

1. ____

2. ____

3. ____

4. ____

5. ____
   - γ is an upper bound of E.
   - γ is least: any $x < \gamma$ is not an upper bound. Find $e \in E$ such that $s + x < e$. OR
   - γ is least: any upper bound β for E satsifies $\gamma \leq \beta$.

6. ____

7. ____

8. ____

9. ____

Definition 0.1. The **extended reals** are defined to be $\mathbb{R} \cup \{\pm\infty\}$. We denote the extended reals as $\overline{\mathbb{R}}$.

Define

$$x + \infty = \infty$$

So

Remark 0.2. In $\overline{\mathbb{R}}$ every subset has a supremum, possibly ∞. eg $\sup E = \infty$. This means E is not bounded.

Let $\vec{x} = (x_1, x_2, ..., x_k)$.

Definition 0.3. Euclidean k space is defined to be

$$\mathbb{R}^k = \{\vec{x} : x_i \in \mathbb{R}\}.$$

We can define addition:

We can define scalar multiplication:

$\mathbb{R}^k$ has an inner product (generalization of the dot product) defined to be:

$< \vec{x}, \vec{y} >= \vec{x} \cdot \vec{y} =$

This multiplication outputs a scalar.

Remark 0.4. We can use the inner product to determine which vectors are perpendicular.

We can also use it to define a norm or length on Euclidean space:

$$|\vec{x}| = (\vec{x} \cdot \vec{x})^{1/2} = \sum_{i=1}^{k} x_i^{2\,1/2}.$$

Aside:

- Addition and scalar multiplication satisfy

- The zero element is $\vec{0} = (0, ..., 0)$.

- We saw above that $\mathbb{R}^k$ has an inner product and a norm. This is not a field as 'obvious' multiplication gives zero divisors which are not allowed since all fields are integral domains. (Abstract Algebra).

Theorem 0.5. *Suppose* $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^k$ *and* $\alpha \in \mathbb{R}$. *Then*

1. $|\vec{x}| \geq 0$.
2. $|\vec{x}| = 0$ *iff* $\vec{x} = \vec{0}$.
3. $|\alpha\vec{x}| = |\alpha||\vec{x}|$.
4. $|\vec{x} \cdot \vec{y}| \leq |\vec{x}||\vec{y}|$
5. $|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$. *Triangle inequality.*
6. $|\vec{x} - \vec{z}| \leq |\vec{x} - \vec{y}| + |\vec{y} - \vec{z}|$

Practice: Try to prove some of these facts!

Geometrically we visualize $\mathbb{R}$ as a line and $\mathbb{R}^2$ as a plane, $\mathbb{R}^3$ (and above) as a hyper plane.

1 The complex numbers and their properties

Let's consider $\mathbb{R}^2$.

$\mathbb{R}^2$ can have an addition and multiplication defined on it such that $\mathbb{R}^2$ is a field, even though in general $\mathbb{R}^k$ is not.

Addition:

$$(a, b) + (c, d) = (a + c, b + d)$$

Multiplication:

What have we defined?

Theorem 1.1. $\mathbb{C}$ *is a field.*

Proof. Check: $(0, 0)$ is the additive identity and $(1, 0)$ is the multiplicative identity.

□

The subset $\{(a, 0) : a \in \mathbb{R}\}$ behaves like $\mathbb{R}$. So $\mathbb{R} \subset \mathbb{C}$ with $\mathbb{R} \to \mathbb{C}$ via:

Via multiplication we can see:

Cool Fact:

$\mathbb{C}$ is **algebraically closed**. This means every non constant polynomial in $\mathbb{C}$ has roots in $\mathbb{C}$. (Fundamental Theorem of Algebra)

Definition 1.2. If $z = a + bi \in \mathbb{C}$ then the **conjugate** is $\bar{z} = a - bi$. We say

$$Re(z) = a$$

and

$$Im(z) = b.$$

Please check for $z, w \in \mathbb{C}$ the following hold:

- $z + \bar{z} = 2Re(z)$
- $\overline{z + w} = \bar{z} + \bar{w}$
- $\overline{zw} = \bar{z}\bar{w}$
- $z\bar{z} = a^2 + b^2 \geq 0.$

Definition 1.3. We define $|z| = (z\bar{z})^{1/2}$ to be the **absolute value** of z. Thus we have a distance measure on $\mathbb{C}$.

Please check for all $z, w \in \mathbb{C}$

- $|z| \geq 0$
- $|\bar{z}| = |z|$
- $|zw| = |z||w|$.
- $Re(z) \leq |z|$.
- No order can be defined on $\mathbb{C}$ such that $\mathbb{C}$ is an ordered field. (Homework problem)
- $|z + w| \leq |z| + |w|$.

Proof. We consider

□

You can also create a **complex** vector space

$$\mathbb{C}^k := \{(z_1, ..., z_k) : z_i \in \mathbb{C}\}.$$

$\mathbb{C}^k$ has an inner product: For $\vec{a}, \vec{b} \in \mathbb{C}^k$ define

$$< \vec{a}, \vec{b} >= \underline{\hspace{6cm}}$$

This inner product ensures that $< \vec{a}, \vec{a} >$ is real and ≥ 0. We can define

$$|\vec{a}| =< \vec{a}, \vec{a} >^{1/2} .$$

Theorem 1.4. *The Cauchy-Schwarz Inequality*
For $\vec{a}, \vec{b} \in \mathbb{C}^k$

$$| < \vec{a}, \vec{b} > |^2 \leq < \vec{a}, \vec{a} >< \vec{b}, \vec{b} >$$

or equivalently

$$|\sum_{i=1}^{k} a_i \bar{b}_i|^2 \leq \sum_{i=1}^{k} |a_i|^2 \sum_{i=1}^{k} |b_i|^2$$

This is the basis for the Heisenberg uncertainty principle, ask your nearest physicist!

Proof. We will prove this inequality for $\mathbb{R}^k$ first; Innovative piece:
Consider $P(x) = \sum_{i=1}^{k}(a_i - xb_i)^2 \geq 0$.

The above proof motivates how we'll prove C-S for $\mathbb{C}^k$;

For an alternative proof see Rudin page 15. □

Example 1.5. Here are some fun consequences of Cauchy Schwarz! I found some from a math olympiad hand book.

- You can use C-S to show that $(x_1 + x_2 + ... + x_n)^2 \le n(x_1^2 + x_2^2 + ... + x_n^2)$.
- Can also use it to show for all positive numbers $a, b, c, d \in \mathbb{R}$ that $16 \le (a+b+c+d)(1/a + 1/b + 1/c + 1/d)$.
- Consider the function $f(x) = \frac{(x+k)^2}{x^2+1}, x \in (-\infty, \infty)$, where k is a positive integer. Show that $f(x) \le k^2 + 1$.
- Let $a_1,\ a_2,\ \cdots,\ a_n,\ b_1,\ b_2,\ \cdots,\ b_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$. Show that

$$\frac{a_1^2}{a_1+b_1} + \frac{a_2^2}{a_2+b_2} + \cdots + \frac{a_n^2}{a_n+b_n} \ge \frac{a_1+a_2+\cdots+a_n}{2}$$

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