Lecture 6–Sept 18: Induction and critiquing proofs

Learning Goals

- Define the well ordering principle (WOP) and the principle of induction (POI).
- Be able to use WOP to prove POI and vice versa.
- Be able to use induction and strong induction to prove theorems.
- Be able to read someone else's work or proof and be able to critique it to help the writer improve.
- Recognize pitfalls in induction proofs.

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1 Induction

Definition 1.1. The well ordering property (WOP) of $\mathbb{N}$.
Any nonempty subset of $\mathbb{N}$ has a least element. We say $\mathbb{N}$ is **well ordered.**

Definition 1.2. Principle of Induction (POI)
Let S be a subset of $\mathbb{N}$ such that

1. $1 \in S$
2. If $k \in S$ then $k + 1 \in S$.

Then $S = \mathbb{N}$.
We say S is an **inductive set.**

Theorem 1.3. *The Well ordering principle implies the principle of induction.*

Proof. Prove this by contradiction.

□

You will show in homework POI $\implies$ WOP.

Recall how to do a proof by induction:
Let $P(n)$ be a statement indexed by $n \in \mathbb{N}$. We will show $P(n)$ is true for all n.

1. Show $P(1)$ is true.
2. Show if $P(k)$ is true then $P(k+1)$ is true.

Then POI shows that $P(n)$ is true for all n.
For strong induction we would instead show if $P(1), ..., P(k)$ is true then $P(k+1)$ is true.

A proof by induction should

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Example 1.4. Prove that for all $n \geq 14$ than n is a sum of 3s and 8s.

Proof. We will show by induction on n that for all $n \geq 14$ $n = \sum 3 + \sum 8$.
Our base case is

□

The above theorem is related to a question in number theory: Question: Given p, q what is the largest number you can not make as a sum of p's and q's? (Frobenius problem)

2 Proof and work critiquing activity

We will discuss your answers as a class after the jigsaw. Ask a member of your group to share out a thought from each problem and we will collect our ideas together here:

Conclusions from the activity:

3 Group 1

In your group discuss the following theorems and their proofs for 5-7 minutes. Discuss: What errors might they be making? What would you need to clarify in these proofs or student work? If the theorem is true, how can you fix your classmates work?

Theorem 3.1. *All natural numbers are even.*

Proof. Suppose all numbers less than or equal to n are even. Notice that
$n + 1 = (n - 1) + 2$ =even +even=even. Hence $n + 1$ is even. □

Theorem 3.2. *All natural numbers are odd.*

Proof. For $n = 1$, 1 is an odd number, so the base case is proved. Suppose all numbers less than or equal to n are odd. Notice that
$n + 1 = (n - 1) + 2$ =odd +even=odd since by the inductive hypothesis $n - 1$ is odd. Hence $n + 1$ is odd. □

Theorem 3.3. *The sum of an odd integer and an even integer is odd.*

Proof. Let a be an odd integer and b an even integer. Then a equals 1+ an even integer, so $a + b$ equals an even integer plus an even integer plus 1. Thus $a + b$ is not even, so it must be odd. □

4 Group 2

In your group discuss the following theorems and their proofs for 5-7 minutes. Discuss: What errors might they be making? What would you need to clarify in these proofs or student work? If the theorem is true, how can you fix your classmates work?

Theorem 4.1. *All cats are the same color.*

Proof. The statement all cats are the same color will be $S(n)$. Consider the set of 1 cat. That cat is the same color as itself, hence the base case $S(1)$ is proven.

We will assume that $S(n)$ is true. Then $S(n+1) = S(n) + 1$ where our inductive hypothesis says our set of n cats are the same color and our base case says that our set of 1 cat is the same color, so our set of $n+1$ cats are the same color. So by POI all cats are the same color. □

The following is related to a homework problem; I skipped proving the base case, as I want you to do this for homework. For the following proof, find the logical error the person is making. **Please don't prove the theorem.**

Theorem 4.2. *A nonempty finite set A in $\mathbb{R}$ contains its supremum.*

Proof. We will prove this by induction on the size of A.

For the base case, if $A = \{a\}$ show $\sup A = a$. [justify this]

Assume our claim holds for any k element set.

Then if A is a k-element set then $B = A \cup \{a\}$ is a $(k+1)$-element set. Since the inductive hypothesis holds for A, it has a supremum, $b \in A$. Either $a < b$, then b is the supremum of B or $b < a$, in which case a is the supremum of B. Thus by POI, we are done. □

5 Group 3

In your group discuss the following theorems and their proofs for 5-7 minutes. Discuss: What errors might they be making? What would you need to clarify in these proofs or student work? If the theorem is true, how can you fix this person's work?

Theorem 5.1. *Every tiling of a $2n \times 2n$ checkerboard using dominoes must have either two 2×1 or two 1×2 dominoes in one corner.*

Proof. For the base case we show all tilings of a 2×2 checkerboard by using two 2×1 dominoes and 1×2 dominoes. See figure A. So the base case is true.

We assume every tiling of a $2n \times 2n$ checkerboard using dominoes must have either two 2×1 or two 1×2 dominoes in one corner.

Now consider a $2(n+1) \times 2(n+1)$ checkerboard. We can create this checkerboard by starting with a $2n \times 2n$ checkerboard and add in the rows and columns as in figure B. By the inductive hypothesis we have two 2×1 or two 1×2 dominoes in one corner of our $2n \times 2n$ checkerboard, so our new $2(n+1) \times 2(n+1)$ checkerboard also has this property. By POI we are done. □

6 Group 4

In your group discuss the following problems for 5-7 minutes. Below are examples of a student's response to some problems in analysis relating to sets. Discuss: Help the student to understand the topic. What would you need to clarify for the student? How would you do that?

1. Let $A = \{1, \{2, 3\}\}$. Decide whether each of the following statements is true or false: $1 \in A$, $1 \subset A$, $\{1\} \in A$, $\{1\} \subset A$, $\{2, 3\} \in A$, $\{2, 3\} \subset A$.

   Student response: T,F,F,T,F,T.

2. Let $X = \{n^2 + 3n - 50 : n \in \mathbb{N}\}$. Find three distinct elements of X.

   Student response: Using $n = 1, 2$, and 10, your classmate computes $n^2 + 3n - 50$, getting $-46, -40$, and 80. They then conclude that $80 \in X$ discarding -46 and -40 as possible answers.

7 Group 5

In your group discuss the following theorems and their proofs for 5-7 minutes. Discuss: What is Theorem 7.1? How could this proof be improved?

Theorem 7.1. *Let $x \in \mathbb{Z}$. If ????????????? then ???????????????.*

Proof. Assume that x is even. Then $x = 2a$ for some integer a. Thus

$$3x^2 - 4x - 5 = 3(2a)^2 - 4(2a) - 5 = 12a^2 - 8a - 5 = 2(6a^2 - 4a - 3) + 1.$$

Since $6a^2 - 4a - 3 \in \mathbb{Z}$, it follows that $3x^2 - 4x - 5$ is odd. □

1. What is Theorem 7.1?
   (a) If x is odd, then $3x^2 - 4x - 5$ is even.
   (b) If $3x^2 - 4x - 5$ is even, then x is odd.
   (c) x is odd only if $3x^2 - 4x - 5$ is even.
   (d) Something else.
2. How could this proof be improved?

8 Example of needing Strong induction

Theorem 8.1. *The fundamental Theorem of arithmetic*

Any integer $n \geq 2$ is either a prime or can be represented as a product of not necessarily distinct primes:

$$n = p_1 p_2 ... p_r, \ p_i \text{ prime.}$$

Proof. We will prove this using strong induction that for all $n \in \mathbb{Z}_{\geq 2}$ $n = p_1 ... p_r$. Call this statement $P(n)$.

□

Remark 8.2. strong induction and induction are equivalent. Sometimes it's cleaner to use 'weak' induction and sometimes, as the above shows, stating strong induction is necessary to prove our statement. Sometimes all we need is weak induction, but we need to prove multiple base cases, ie Fibonacci style problems.