Lecture 7–Sept 20: Countability and Uncountability

Learning Goals

- Learn how to count. Be able to formalize counting using $\mathbb{N}$ and functions.
- Be able to use induciton to prove $\mathbb{N}$ is infinite.
- Define what it means for a set to be countable or uncountable.
- Be able to prove $\mathbb{Z}$, $\mathbb{Q}$, etc are countable.
- Be able to prove if $\mathbb{R}$ is countable or uncountable.
- Be able to use arrays to prove countability and Cantor diagonlization arguments to show uncountability.
- Define cardinality and power sets.

FRIDAY September 22: HW 3 DUE at **12pm**

Let's dive into chapter 2 of Rudin.

1 Counting and countability

Question: How do we count?

Answer: ____________________

Recall: $f : A \to B$ (Domain into Codomain) associates

Some definitions:

- We write $f(C) = \{f(x) : x \in C\}$, ____________________
- $f^{-1}(D) = \{x : f(x) \in D\}$ ____________________
- When $f(A) =$ the entirety of B we say f is **onto**. or surjective: $\twoheadrightarrow$
- When $f(x) = f(y)$ implies $x = y$, we say f is **one to one** or injective. $\hookrightarrow$
- When f is one to one and onto ____________________.

A bijection puts A and B into a 'one to one correspondence.' We write $A \sim B$. Please check that this is an equivalence relation.

Example 1.1. An example of a bijection.

Definition 1.2. We say A is finite if $A \sim \{1, 2, ..., n\}$ for some $n \in \mathbb{N}$. We say A has cardinality n and denote it $[n]$. Or A is empty, $\emptyset$.

Else we say A is infinite.

Definition 1.3. We say an infinite set A is countable if $A \sim \mathbb{N}$. (Here think $A = \{a_1, a_2, ...\}$ has a bijection into $\mathbb{N}$). Else an infinite set A is uncountable.

Example 1.4.

- $\mathbb{N}$

- *Definition* 1.5. A **sequence** is a function f defined on the set $\mathbb{N}$. If $f(n) = x_n$, for $n \in \mathbb{N}$ it is customary to denote sequence f by $\{x_n\} = x_1, x_2, ...$ The values of f are terms of the sequence.

  So a sequence $x_1, ...x_n, ...$ of distinct terms

- $\{2, 3, 4...\}$

- $\mathbb{N} \setminus \{k\}$

Theorem 1.6. $\mathbb{N}$ *is infinite.*

Proof. We'll show there does not exist a bijection $[n] \to \mathbb{N}$ by induction on n.

$\square$

Example 1.7.

- $2\mathbb{N}$ (the even numbers)

- $\mathbb{Z}$ is countable

- Hilberts Hotel: Hilbert's hotel has infinitely many rooms and it has a no vacancy sign up. How can the manager rearrange their customers so that they have vacancies?

Theorem 1.8. *Every infinite subset of countable sets is countable.*

Proof. sketch Let $E \subset A$, where A is a countable set.

□

Remark 1.9. This means that no uncountable set can be a subset of a countable set. Countable sets are the 'smallest' infinite set.

Theorem 1.10. $\mathbb{Q}$ *is countable.*

Proof. We need to list $\mathbb{Q}$ as a sequence, then that sequence can be in one to one correspondence with $\mathbb{N}$. Let's first do $\mathbb{Q}^+$ then we will use the trick we used for $\mathbb{Z} \sim \mathbb{N}$ to get all of $\mathbb{Q}$.

Why is repetition a non issue?

□

Theorem 1.11. *An (at most) countable union of (at most) countable sets is (at most) countable.*

Proof. The array above will help us create a countable set. If we have countable sets $A_1, A_2, ...$ then we can create an array of their elements.

□

Question: Is $\mathbb{R}$ countable?

If $\mathbb{R}$ is not countable then there should be some real number that can't have a decimal expansion produced by a computer program.

Guesses?

Theorem 1.12 (Due to Cantor 1874.)**.** $\mathbb{R}$ *is* ________________.

Proof. It is enough to consider the subest $[0, 1) \subset \mathbb{R}$.

Things to be careful: Do not always pick 9 as your alternative number since $1 = .99999...$ □

Definition 1.13. Give a set A we let 2^A denote the **power set** of A. The power set is the set of all subsets of A, include A and $\emptyset$.

Example 1.14. An example of a power set:

Definition 1.15. We say A and B have the same cardinality if and only if $A \sim B$.

Theorem 1.16 (Due to Cantor 1891.)**.** *For any set A we have that* $A \not\sim 2^A$.

This theorem says A and 2^A have different cardinalities. In fact when we consider $\mathbb{N}$, we see that the power set $2^{\mathbb{N}} \sim \mathbb{R}$.

The continuum hypothesis (independent axiom of set theory) states that the first cardinality bigger than $\mathbb{N}$ is $\mathbb{R}$.