Lecture 8–Sept 25: Metric Spaces

Learning Goals

- Be able to define a metric space.
- Be able to determine if a space is a metric space.
- Define open and closed sets, interior points, limit points, and isolated points.
- Be able to use these definitions to show that a set is open or closed.

---

1 Metric Spaces

We want to know how do we define distances between points, between functions, or between genome sequences, etc?

One way is to create a metric space.

Definition 1.1. A set X is a **metric space** if there exists a metric

$$d : X \times X \to \mathbb{R}$$

such that for all $p,\ q \in X$

1. $d(p, q) \geq 0$ and $d(p, q) = 0$ iff $p = q$,
2. $d(p, q) = d(q, p)$, and
3. $d(p, q) \leq d(p, w) + d(w, q)$ for all $w \in X$.

We say (X, d) is a metric space.

Remark 1.2. If (X, d) is a metric space then so is $Y \subset X$ when equipped with the same metric.

Example 1.3. Examples of metric spaces:

1. $\mathbb{R}$ or $\mathbb{C}$ with $d(x, y) = |x - y|$.
2. $\mathbb{R}^n$ with $d(\vec{x}, \vec{y}) = ||\vec{x} - \vec{y}||$ with $||\vec{w}|| = \sqrt{w_1^2 + w_2^2 + ... + w_n^2}$.
3. $\mathbb{R}^n$ with $d(\vec{x}, \vec{y}) = \sum_{i=1}^{n} |x_i - y_i|$.

4. $\mathbb{R}^n$ with the L^p metric $d(\vec{x}, \vec{y}) = (\sum_{i=1}^n |x_i - y_i|^p)^{1/p}$.

5. Let X be the tree graph and $d(x, y) =$ the length of the shortest path. Is this a metric?

   Question: Given $d(x_i, x_j)$ is there a tree that produces the same distances?

6. $X = \{$ genome sequence$\}$ then $d(x, y) =$ the number of letters that differ.

7. $X = \{$continuous functions from $\mathbb{R} \to \mathbb{R}\}$.

8. $\mathbb{R}^n$ with entropy distance $d(\vec{p}, \vec{q}) = \sum P_i \log(\frac{p_i}{q_i})$. Is this a metric space? Why?

9. $\mathbb{Q}$ has the p-adic distance $d(x,y) = |x-y|_p$. Let p be a prime. If $\frac{a}{b} = p^r \frac{a'}{b'}$ with $p \nmid a',\ b'$ then $|\frac{a}{b}|_p = p^{-r}$.

Now that we have a way of measuring distances we can talk about what it means for a general set to be bounded.

Definition 1.4. E is **bounded** if there exists $M \in \mathbb{R}$ and a point $q \in X$ such that $d(p,q) < M$ for all $p \in E$.

2 The topology of $\mathbb{R}^n$

We want to know what elements are close by. So we define the following.

Definition 2.1. We define the **ball** or **neighborhood** N_r or $B(x,r)$ to be

$$N_r(x) = \{y : d(x,y) < r\}.$$

Example 2.2. Some examples of a neighborhood:

The moral is that a neighborhood or ball tells us what's close in a metric space.

Definition 2.3. A point $p \in X$ is a **limit point** of E if every neighborhood of p contains a point $q \in E$, where $q \neq p$. (p does not necessarily have to be in E.)

If $p \in E$ and p is not a limit point of E then we say p is an **isolated point.**

Example 2.4.

- Let our set be $\mathbb{R}$ equipped with the usual metric ($|x-y|$) and define

$$E = \{\frac{1}{n} : n \in \mathbb{Z}\}.$$

- Let $\mathbb{R}$ have the discrete metric with the same E as above.

The discrete metric will often be a counter example to statements in this class.

Definition 2.5. We say p is an **interior point** of E if there exists a neighborhood N of p such that $N \subset E$.

Definition 2.6. A set E is **open** if every point is an interior point.

Question: Are balls open?
Guesses?

Theorem 2.7. *Neighborhoods of a point are* __________________.

Proof. Given $B(x, r)$ we will show $p \in B(x, r)$ is

□

Definition 2.8. A set K is **closed** if k contains all its limit points.

Remark 2.9. Note a set can be neither open nor closed. It can also be both open an closed.

Example 2.10. Examples of closed sets, sets that are neither open or closed, and sets that are both.

- $E = \{\frac{1}{n} : n \in \mathbb{Z}\} \subset \mathbb{R}$ equipped with the usual metric

  If we equip $\mathbb{R}$ with the discrete metric, then E is

- $\mathbb{Z} \subset \mathbb{R}$ with the usual metric.

- Consider

- Buzzwords: In a complete nonarchimedean space neighborhoods are both open and closed.

Theorem 2.11. *If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.*

Proof. Proof by contradiction.

□

Definition 2.12. E is dense in X if every point of X is a limit point of E or a point of E or both.

Example 2.13.

- $\mathbb{Q}$ is dense in $\mathbb{R}$.

- $\mathbb{Z}$ is

3 Fun facts

Definition 3.1. A set $E \subset \mathbb{R}^k$ is **convex** if $\lambda\vec{x} + (1-\lambda)\vec{y} \in E$ whenever $\vec{x},\ \vec{y} \in E$ and $0 < \lambda < 1$.

Theorem 3.2. *Balls are convex.*