Lecture 9–Sept 27: Open and closed sets

Learning Goals

- Be able to define what it means to be open or closed.
- Be able to use the definition of open and closed to prove a set is either open or closed.
- Be able to prove equivalent statements to what it means to be closed or open.
- Use equivalent statement to show a set is open or closed.
- State and prove DeMorgan's Law.

Exam covers Lectures 1-9. HW 1-4 and Rudin page 1-36.
Hw 4 due Friday at noon. No Homework next week.

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1 Open and closed sets

Recall:

Definition 1.1. A set E is **open** if every point is an interior point.

Theorem 1.2. *Balls are open.*

Proof. Given $B(x, r)$ we will show $p \in B(x, r)$ is interior. (nothing special about this particular p so it will work for all points in the ball).

Let

□

Definition 1.3. A set K is **closed** if k contains all its limit points.

Remark 1.4. Note a set can be neither open nor closed. It can also be both open an closed.

Example 1.5. Examples of open and closed sets.

1. Some examples of open and closed sets we may be familiar with:

2. $\emptyset$

3. Let $\mathbb{R}$ be our universe. Then the subset $\mathbb{R} \subset \mathbb{R}$ is

   We say $\mathbb{R}$ and $\emptyset$ are **clopen.** This is true for $\mathbb{R}$ with any metric.

4. Suppose $\mathbb{R}$ has the discrete metric.

5. $E = \{\frac{1}{n} : n \in \mathbb{Z}\} \subset \mathbb{R}$ equipped with the usual metric.

6. $\mathbb{Z} \subset \mathbb{R}$ with the usual metric.

7. Number theory words: In a complete nonarchimedean space neighborhoods are both open and closed.

Theorem 1.6. *If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.*

Proof. Proof by contradiction.

□

Definition 1.7. E is dense in X if every point of X is a limit point of E or a point of E or both.

Example 1.8. Examples of dense and not dense sets.

- $\mathbb{Q}$
- $\mathbb{Z}$

Example 1.9. Logic practice: Here are examples of equivalent statements. We should practice moving between for all statements and there exists statements.

- A set is *open* if *every point* is an *interior point.*
- A point p is a limit point of E if every neighborhood of p contains a $q \in E$ where $q \neq p$.

These equivalent statement will help us prove theorems we're interested in.

Definition 1.10. Let E' be the set of all limit points of E. The **closure** of E is $\overline{E} = E \cup E'$.

This leads to the question: Is $\overline{E}$ closed?

If it wasn't closed, calling it the closure would be a terrible name! But having a name that suggests it's closed is not a proof.

Theorem 1.11. *The closure is closed.*

Proof. To show $\overline{E}$ is closed we need to show that $\overline{E}$ contains all of its limit points, as per the definition of closed.

□

Theorem 1.12. *E is closed if and only if $E = \overline{E}$.*

Proof. First we

□

Theorem 1.13. *If $E \subset F$, where F is a closed set, then $\overline{E} \subset F$.*

Remark 1.14. These two theorems says that $\overline{E}$ is the 'smallest' closed set containing the set E.

Proof. Proof of Theorem 1.13. We know $E \subset F$, to show $\overline{E} \subset F$ we need to show that $E' \subset F$. □

Theorem 1.15. *E is open if and only if E^c is closed.*

Recall that $E^c = \{p : p \in X \setminus E\}$, the complement of E in the universal set X.

Proof. We will write down equivalent statements to being open and arrive at the fact that E^c is closed. □

Corollary 1.16. *A set F is closed if and only if its complement is open.*

Theorem 1.17. *Let* $\{E_\alpha\}_{\alpha\in\lambda}$ *be a (finite or infinite) collection of sets* E_α*. Then*

$$(\bigcup_{\alpha\in\lambda} E_\alpha)^c = \bigcap_{\alpha\in\lambda} (E_\alpha)^c.$$

This is often referred to as ***DeMorgan's Law.***

Proof. We want to show that if $x \in (\bigcup_{\alpha\in\lambda} E_\alpha)^c$ then $x \in \bigcap_{\alpha\in\lambda}(E_\alpha)^c$. And we want to show that the reverse is true.

Now you try the reverse direction.

□

Short aside:

Definition 1.18. A set $E \subset \mathbb{R}^k$ is **convex** if $\lambda\vec{x} + (1-\lambda)\vec{y} \in E$ whenever $\vec{x},\ \vec{y} \in E$ and $0 < \lambda < 1$.

Example 1.19. Balls are convex.

end of aside.